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**An introduction to
Tauberian theory:
from Tauber to Wiener**

J. van de Lune



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PREFACE

Although Tauberian theory actually is still a living part of mathematics, it appears that more and more students are at best remotely familiar with this topic which was so popular in the first half of this century.

By writing this booklet I have tried to present an easily readable, fairly detailed sketch of the early development of classical Tauberian theory: a "continuous" mathematical history, from Alfred Tauber to Norbert Wiener.

I hope that in this booklet the reader will find a clear implicit answer to the question "What is a Tauberian theorem?", and that (s)he will find the enthusiasm to pursue the subject in the great variety of directions I have failed to mention.

J. van de Lune

CONTENTS

CHAPTER 1. THE ORIGIN OF TAUBERIAN THEORY	1
0. Introduction	1
1. Tauber's first theorem	2
2. Tauber's second theorem	4
3. A theorem of Fejér and its generalization	7
CHAPTER 2. THE O -THEOREMS OF HARDY AND LITTLEWOOD	11
1. The O -theorems of Hardy and Littlewood	11
2. Some extensions to general Dirichlet-series	19
CHAPTER 3. SOME APPLICATIONS	21
1. The series $\sum_{n=1}^{\infty} n^{-1-it}$ is not Abel-summable	21
2. An alternative proof of a theorem of Jordan	22
CHAPTER 4. INTRODUCTION TO PITT'S GENERAL TAUBERIAN THEOREM	24
CHAPTER 5. INTRODUCTION TO WIENER'S GENERAL TAUBERIAN THEOREM	28
1. First approach	28
2. Second approach	32
CHAPTER 6. FOURIER TRANSFORMS	33
1. The simplest properties of Fourier transforms	33
2. Fourier-inversion (Recovering f from \hat{f})	36
3. Convolution products of functions in L^1	43
4. Approximate identities	44
5. Existence and construction of certain Fourier transforms	47
CHAPTER 7. ANALYTIC FUNCTIONS OF FOURIER TRANSFORMS	52
CHAPTER 8. WIENER'S GENERAL TAUBERIAN THEOREM	59
1. Some simple properties of T_f and \bar{T}_f	59
2. Proof of Wiener's theorem	59
CHAPTER 9. SOME ALGEBRAIC REFORMULATIONS	64
CHAPTER 10. SOME ANALYTIC REFORMULATIONS	69
CHAPTER 11. PITT'S GENERAL TAUBERIAN THEOREM	73
1. Pitt's theorem for slowly oscillating functions on \mathbb{R}	73
2. Pitt's theorem for slowly oscillating functions on \mathbb{R}^+	77

CHAPTER 11. (continued)	
3. Some applications	78
CHAPTER 12. A RELATED TOPIC: CLOSED SYSTEMS	80
CHAPTER 13. IKEHARA'S THEOREM	88
0. Introduction	88
1. Ikehara's theorem	89
2. Bochner's theorem	93
CHAPTER 14. THE PRIME NUMBER THEOREM	94
1. Number-theoretical preliminaries	94
2. Some fundamental facts about $\zeta(s)$	98
3. The prime number theorem	101

CHAPTER 1

THE ORIGIN OF TAUBERIAN THEORY

0. INTRODUCTION

We begin our considerations by recalling a more or less standard version of Abel's limit theorem for power series.

THEOREM 0.0. (1826, ABEL [1]) *If the power series*

$$(0.1) \quad \sum_{k=0}^{\infty} a_k x^k$$

with complex coefficients, converges for $x = 1$, then

$$(0.2) \quad \lim_{x \uparrow 1} f(x) \text{ exists and } = f(1) = \sum_{k=0}^{\infty} a_k \text{ (= A, say)}$$

where $f(x)$ denotes the sum of (0.1) for $-1 < x \leq 1$.

Briefly formulated we thus have

$$(0.3) \quad \sum_{k=0}^{\infty} a_k = A \Rightarrow \lim_{x \uparrow 1} f(x) = A.$$

For a proof of this celebrated theorem we refer to KNOPP [3; pp. 179-180] or TITCHMARSH [7; pp. 9-10].

One may ask (as Tauber did) whether the converse of Abel's theorem, i.e. the statement

$$(0.4) \quad \lim_{x \uparrow 1} f(x) = A \Rightarrow \sum_{k=0}^{\infty} a_k = A$$

also holds true.

That this is *not* the case in general may be shown by the following simple example. Let $a_k := (-1)^k$ for $k = 0, 1, 2, 3, \dots$; then $f(x) = (1+x)^{-1}$ for $|x| < 1$, so that $\lim_{x \uparrow 1} f(x) = 1/2$, whereas $\sum_{k=0}^{\infty} a_k$ is clearly divergent.

It follows that the converse of Abel's theorem can only be true if we impose some additional (so called Tauberian) condition(s).

As a first example we mention the following very simple Tauberian

THEOREM 0.1. *If $a_k \geq 0$ for all sufficiently large k , then the converse of Abel's theorem holds true.*

PROOF. Since $a_k \geq 0$ for all sufficiently large k , we have for all sufficiently large n

$$(0.5) \quad S_n := \sum_{k=0}^n a_k = \lim_{x \uparrow 1} \sum_{k=0}^n a_k x^k \leq \lim_{x \uparrow 1} f(x) = A$$

so that the eventually monotone non-decreasing sequence S_n is bounded and hence convergent. It follows that $\sum_{k=0}^{\infty} a_k$ is convergent so that we may complete the proof by simply invoking Abel's limit theorem.

However, the use of Abel's theorem may be avoided here as follows. For $0 < x < 1$, and n sufficiently large, we have

$$(0.6) \quad |S_n - f(x)| \leq \sum_{k=1}^n |a_k| (1-x^k) + \sum_{k=n+1}^{\infty} a_k$$

so that (by taking limits for $x \uparrow 1$)

$$(0.7) \quad |S_n - A| \leq \sum_{k=n+1}^{\infty} a_k.$$

Since the right hand side is the "general tail" of a convergent series it follows that $\lim_{n \rightarrow \infty} S_n = A$. \square

1. TAUBER'S FIRST THEOREM

The first non-trivial theorem, establishing the convergence of $\sum_{k=0}^{\infty} a_k$ from the behaviour of the sum of its associated power series *and some additional condition on the terms a_k* , was given by Tauber.

THEOREM 1.1. (1897, TAUBER [6]) *If $\lim_{k \rightarrow \infty} k a_k = 0$ or, equivalently, if*

$$(1.1) \quad a_k = o\left(\frac{1}{k}\right), \quad (k \rightarrow \infty)$$

then the converse of Abel's theorem holds true.

FIRST PROOF. See TAUBER [6] or KNOPP [3; pp. 518-519].

SECOND PROOF. For $0 < x < 1$ we have, with $\varepsilon_n := \sup_{k > n} \{k |a_k|\}$,

$$(1.2) \quad \begin{aligned} |S_n - f(x)| &\leq \sum_{k=1}^n |a_k| (1-x^k) + \sum_{k=n+1}^{\infty} |a_k| x^k = \\ &= (1-x) \sum_{k=1}^n |a_k| (1+x+\dots+x^{k-1}) + \sum_{k=n+1}^{\infty} k |a_k| \frac{x^k}{k} \leq \end{aligned}$$

$$\begin{aligned}
 &\leq (1-x) \sum_{k=1}^n k |a_k| + \frac{\varepsilon_n}{n+1} \cdot \sum_{k=n+1}^{\infty} x^k \leq \\
 (1.2a) \quad &\leq (1-x)n\varepsilon_0 + \frac{\varepsilon_n}{n+1} \frac{1}{1-x}.
 \end{aligned}$$

In order to avoid trivialities we assume that all ε_n are positive. We try to choose $x = x_n$ such that (1.2a) is minimal. In order to do so we observe that the function

$$(1.3) \quad \phi(t) := at + \frac{b}{t}, \quad (t > 0)$$

where a and b are positive constants, assumes its minimal value $2(ab)^{1/2}$ at the point $t = (\frac{b}{a})^{1/2}$. In particular, with $a = n\varepsilon_0$ and $b = \varepsilon_n/(n+1)$, we find that (1.2a) assumes its minimal value

$$(1.4) \quad 2\left\{\frac{n\varepsilon_0\varepsilon_n}{n+1}\right\}^{1/2}$$

at the point $x = x_n$ defined by

$$(1.5) \quad (0 <) 1 - x_n = \left\{\frac{\varepsilon_n}{n(n+1)\varepsilon_0}\right\}^{1/2} (< 1).$$

It follows that

$$(1.6) \quad |S_n - f(x_n)| \leq 2\left\{\frac{n\varepsilon_0\varepsilon_n}{n+1}\right\}^{1/2} < 2(\varepsilon_0\varepsilon_n)^{1/2}.$$

Since (a) x_n tends to 1 from the left as $n \rightarrow \infty$

(b) $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, and

(c) $|S_n - A| \leq |S_n - f(x_n)| + |f(x_n) - A|$

the theorem follows. \square

For some generalizations of Tauber's first theorem see LANDAU [4; §8, §11, §12].

The integral analogue of Tauber's first theorem reads as follows.

THEOREM 1.1A. *If the function $a(t)$ is (Lebesgue) integrable over $[0, T]$ (notation: $a \in L^1[0, T]$) for every $T > 0$, and if $\lim_{t \rightarrow \infty} ta(t) = 0$, then from*

$$(1.7) \quad \lim_{s \rightarrow 0} \int_0^{\infty} e^{-st} a(t) dt = A$$

it follows that

$$(1.8) \quad \int_0^{\rightarrow \infty} a(t) dt := \lim_{T \rightarrow \infty} \int_0^T a(t) dt = A.$$

PROOF. See WIDDER [8; p. 186].

2. TAUBER'S SECOND THEOREM

THEOREM 2.1. (1897, TAUBER [6]) If

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n ka_k = 0$$

then the converse of Abel's theorem holds true.

This is the Tauberian part of the following

THEOREM 2.2. (1897, TAUBER [6]) The series $\sum_{k=0}^{\infty} a_k$ converges with sum A if and only if

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n ka_k = 0$$

and

$$(2.3) \quad \lim_{x \uparrow 1} \sum_{k=0}^{\infty} a_k x^k = A.$$

PROOF.

(\Rightarrow) From the convergence of $\sum_{k=0}^{\infty} a_k$ we may obtain (2.2) by means of a general theorem of Kronecker (see KNOPP [3; p. 131]) or directly by observing that

$$(2.4) \quad \begin{aligned} \frac{1}{n} \sum_{k=1}^n ka_k &= \frac{1}{n} \sum_{k=1}^n k(S_k - S_{k-1}) = \\ &= \frac{1}{n} \{nS_n - (S_1 + S_2 + \dots + S_{n-1})\} = S_n - \frac{S_1 + S_2 + \dots + S_{n-1}}{n-1} \frac{n-1}{n} \end{aligned}$$

which, by Cauchy's limit theorem, tends to 0 as $n \rightarrow \infty$.

Clearly (2.3) follows from Abel's limit theorem.

(\Leftarrow) This is the more elaborate Tauberian part of the theorem. We first note that $\sum_{k=0}^{\infty} a_k x^k$ converges for $|x| < 1$. In order to see this we write $w_0 = 0$, and $w_k = a_1 + 2a_2 + \dots + ka_k$ for $k \geq 1$ so that $w_k = o(k)$ as $k \rightarrow \infty$. Hence, the radius of convergence of the power series $\sum_{k=0}^{\infty} w_k x^k$ is ≥ 1 so that for $|x| < 1$

$$(2.5) \quad (1-x) \sum_{k=0}^{\infty} w_k x^k = \sum_{k=0}^{\infty} w_k x^k - \sum_{k=0}^{\infty} w_k x^{k+1} = \sum_{k=1}^{\infty} ka_k x^k$$

from which it follows that the radius of convergence of $\sum_{k=0}^{\infty} ka_k x^k$

is at least 1. Since the power series $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=1}^{\infty} k a_k x^k$ have the same radius of convergence it follows that $\sum_{k=0}^{\infty} a_k x^k$ converges for $|x| < 1$.

Next we observe that for $|x| < 1$ (using partial summation)

$$\begin{aligned}
 (2.6) \quad \sum_{k=0}^{\infty} a_k x^k - a_0 &= \sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} \frac{k a_k}{k} x^k = \\
 &= \sum_{k=1}^{\infty} \frac{w_k - w_{k-1}}{k} x^k = \sum_{k=1}^{\infty} w_k \left\{ \frac{x^k}{k} - \frac{x^{k+1}}{k+1} \right\} = \\
 &= \sum_{k=1}^{\infty} w_k \left\{ \frac{x^k}{k+1} + \frac{x^k}{k(k+1)} \right\} = \\
 &= (1-x) \sum_{k=1}^{\infty} \frac{w_k}{k+1} x^k + \sum_{k=1}^{\infty} \frac{w_k}{k(k+1)} x^k.
 \end{aligned}$$

We now claim that

$$(2.7) \quad \lim_{x \uparrow 1} (1-x) \sum_{k=1}^{\infty} \frac{w_k}{k+1} x^k = 0.$$

In order to see this we observe that for $0 < x < 1$

$$\begin{aligned}
 (2.8) \quad \left| (1-x) \sum_{k=1}^{\infty} \frac{w_k}{k+1} x^k \right| &\leq (1-x) \left\{ \left| \sum_{k=1}^n \frac{w_k}{k+1} x^k \right| + \right. \\
 &+ \left. \left| \sum_{k=n+1}^{\infty} \frac{w_k}{k+1} x^k \right| \right\} \leq \dots (\delta_n := \sup_{k>n} \frac{|w_k|}{k+1}) \dots \\
 &\leq (1-x) \sum_{k=1}^n \frac{|w_k|}{k+1} + \delta_n.
 \end{aligned}$$

It follows that for every n

$$(2.9) \quad \limsup_{x \uparrow 1} \left| (1-x) \sum_{k=1}^{\infty} \frac{w_k}{k+1} x^k \right| \leq \delta_n$$

and since $\lim_{n \rightarrow \infty} \delta_n = 0$, our claim (2.7) follows.

From (2.6) and (2.7) we thus obtain

$$(2.10) \quad A - a_0 = \lim_{x \uparrow 1} \sum_{k=1}^{\infty} \frac{w_k}{k(k+1)} x^k.$$

Since

$$(2.11) \quad \lim_{k \rightarrow \infty} k \frac{w_k}{k(k+1)} = \lim_{k \rightarrow \infty} \frac{w_k}{k+1} = 0$$

Tauber's first theorem applies to (2.10), so that

$$(2.12) \quad \sum_{k=1}^{\infty} \frac{w_k}{k(k+1)} = A - a_0.$$

Observing that

$$(2.13) \quad \begin{aligned} \sum_{k=1}^{\infty} \frac{w_k}{k(k+1)} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{w_k}{k(k+1)} = \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n w_k \left\{ \frac{1}{k} - \frac{1}{k+1} \right\} = \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n \frac{w_k - w_{k-1}}{k} - \frac{w_n}{n+1} \right\} = \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k - \lim_{n \rightarrow \infty} \frac{w_n}{n+1} = \sum_{k=1}^{\infty} a_k \end{aligned}$$

it follows that $\sum_{k=1}^{\infty} a_k$ converges to $A - a_0$, completing the proof. \square

The integral analogue of Tauber's second theorem reads

THEOREM 2.2A. *Let $\phi \in L^1[0, T]$ for every $T > 0$. Then*

$$(2.14) \quad \int_0^{\rightarrow \infty} \phi(t) dt = A$$

if and only if

$$(2.15) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u \phi(u) du = 0$$

and

$$(2.16) \quad \lim_{s \rightarrow 0} \int_0^{\infty} e^{-st} \phi(t) dt = A.$$

This theorem may be generalized as follows

THEOREM 2.2B. *Let the function $\alpha(t)$ be of bounded variation on every interval $[0, T]$ with $T > 0$. Then*

$$(2.17) \quad \lim_{t \rightarrow \infty} \alpha(t) = A + \alpha(0)$$

if and only if

$$(2.18) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u \, d\alpha(u) = 0$$

and

$$(2.19) \quad \lim_{s \downarrow 0} \int_0^{\infty} e^{-st} \, d\alpha(t) = A.$$

PROOF. See WIDDER [8; pp. 187-188]. \square

(Widder's proof needs slight adjustment !)

As an application we derive the Tauberian part of Theorem 2.2 from Theorem 2.2B.

Define

$$(2.20) \quad \alpha(t) := \sum_{k < t} a_k, \quad (t \geq 0).$$

Then, for $x = e^{-s}$, $s > 0$,

$$(2.21) \quad \sum_{k=0}^{\infty} a_k x^k = \int_0^{\infty} x^t \, d\alpha(t) = \int_0^{\infty} e^{-st} \, d\alpha(t)$$

so that

$$(2.22) \quad \lim_{s \downarrow 0} \int_0^{\infty} e^{-st} \, d\alpha(t) = A.$$

Since, for $t \rightarrow \infty$,

$$(2.23) \quad \frac{1}{t} \int_0^t u \, d\alpha(u) = \frac{1}{t} \sum_{k < t} k a_k \rightarrow 0$$

it follows from Theorem 2.2B that

$$(2.24) \quad \lim_{t \rightarrow \infty} \alpha(t) = A + \alpha(0)$$

so that (note that $\alpha(0) = 0$)

$$(2.25) \quad \lim_{n \rightarrow \infty} \alpha(n+1) = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k = \sum_{k=0}^{\infty} a_k = A.$$

3. A THEOREM OF FEJÉR AND ITS GENERALIZATION

We conclude this chapter by proving another early Tauberian theorem due to Fejér (see LANDAU [4; pp. 59-60]).

THEOREM 3.1. (1913, FEJÉR) If

$$\sum_{k=1}^{\infty} k |a_k|^2 \quad \text{converges and}$$

$$\lim_{x \uparrow 1} \sum_{k=0}^{\infty} a_k x^k = A, \quad \text{then} \quad \sum_{k=0}^{\infty} a_k = A.$$

Hardy and Littlewood generalized this to

THEOREM 3.2. (1914, HARDY & LITTLEWOOD [2]) *If there exists a constant $p > 1$ such that*

$$(3.1) \quad \sum_{k=1}^{\infty} k^{p-1} |a_k|^p$$

converges and if

$$(3.2) \quad \lim_{x \uparrow 1} \sum_{k=0}^{\infty} a_k x^k = A$$

then

$$(3.3) \quad \sum_{k=0}^{\infty} a_k = A.$$

PROOF. In principle the proof is very similar to the second proof of Theorem 1.1. In addition we will make use of Hölder's inequality (see RUDIN [5; p. 62])

$$(3.4) \quad \left| \sum_{k=1}^n u_k v_k \right| \leq \left\{ \sum_{k=1}^n |u_k|^p \right\}^{1/p} \left\{ \sum_{k=1}^n |v_k|^q \right\}^{1/q}$$

where all numbers u_k and v_k are complex, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. From our Tauberian condition it is easily seen (use the Cauchy-Hadamard formula for the radius of convergence of a power series) that $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $|x| < 1$, so that, without any further assumptions, the function

$$(3.5) \quad f(x) := \sum_{k=0}^{\infty} a_k x^k, \quad (|x| < 1)$$

is well defined.

For $0 < x < 1$ we then have (with $\alpha := \frac{p-1}{p} = \frac{1}{q}$)

$$(3.6) \quad \begin{aligned} \left| \sum_{k=0}^n a_k - f(x) \right| &\leq (1-x) \sum_{k=1}^n k |a_k| + \sum_{k=n+1}^{\infty} |a_k| x^k = \\ &= (1-x) \sum_{k=1}^n k^\alpha |a_k| k^{1-\alpha} + \sum_{k=n+1}^{\infty} k^\alpha |a_k| \frac{x^k}{k^\alpha} \leq \\ &\leq (1-x) \left\{ \sum_{k=1}^n (k^\alpha |a_k|)^p \right\}^{1/p} \left\{ \sum_{k=1}^n (k^{1-\alpha})^q \right\}^\alpha + \end{aligned}$$

$$\begin{aligned}
& + \left\{ \sum_{k=n+1}^{\infty} (k^{\alpha} |a_k|)^p \right\}^{1/p} \left\{ \sum_{k=n+1}^{\infty} \left(\frac{x^k}{k^{\alpha}} \right)^q \right\}^{\alpha} \leq \\
& \dots (r_n := \left\{ \sum_{k=n}^{\infty} (k^{\alpha} |a_k|)^p \right\}^{1/p}) \dots \\
& \leq (1-x) r_1 \left\{ \int_0^{n+1} t^{q-1} dt \right\}^{\alpha} + r_{n+1} \frac{1}{(n+1)^{\alpha}} \left\{ \sum_{k=n+1}^{\infty} x^{qk} \right\}^{\alpha} \leq \\
& \leq (1-x) r_1 (n+1)^q^{-\alpha} + \frac{r_{n+1}}{(n+1)^{\alpha}} \frac{1}{(1-x)^{\alpha}} = \\
& = a(1-x) + b(1-x)^{-\alpha}
\end{aligned}$$

where $a = a(n) = r_1 (n+1) q^{-\alpha}$ and $b = b(n) = r_{n+1} (n+1)^{-\alpha}$.
 In order to avoid trivialities we assume that a and b are positive. One may verify that the function

$$(3.7) \quad \phi(t) = at + bt^{-\alpha}, \quad (t > 0)$$

is minimal at $t = \left(\frac{b\alpha}{a} \right)^{\beta}$ where $\beta = \frac{q}{q+1}$.

For our purpose we therefore define x_n by

$$(3.8) \quad 1-x_n = \left\{ \frac{r_{n+1}}{(n+1)^{\alpha}} \frac{\alpha}{r_1} \frac{q^{\alpha}}{n+1} \right\}^{\beta} = \frac{c_0}{n+1} (r_{n+1})^{\beta}$$

the meaning of the constant c_0 being clear from the context. Fortunately x_n lies between 0 and 1 if n is sufficiently large, and x_n tends to 1. The corresponding (minimal) value of ϕ is

$$\begin{aligned}
(3.9) \quad & \frac{c_0}{n+1} (r_{n+1})^{\beta} r_1 (n+1) q^{-\alpha} + \frac{r_{n+1}}{(n+1)^{\alpha}} \left\{ \frac{n+1}{c_0} r_{n+1}^{-\beta} \right\}^{\alpha} = \\
& = c_1 (r_{n+1})^{\beta} + c_2 (r_{n+1})^{\beta} = c_3 (r_{n+1})^{\beta}
\end{aligned}$$

which tends to 0 as $n \rightarrow \infty$. Since x_n tends to 1 from the left as $n \rightarrow \infty$ it follows that

$$(3.10) \quad \lim_{n \rightarrow \infty} \left\{ \sum_{k=0}^n a_k - f(x_n) \right\} = 0$$

so that $\sum_{k=0}^{\infty} a_k = A$, completing the proof. \square

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CHAPTER 2

THE O -THEOREMS OF HARDY AND LITTLEWOOD

The main object of this chapter is to present a considerable improvement of Tauber's first theorem.

1. THE O -THEOREMS OF HARDY AND LITTLEWOOD

THEOREM 1.1. (1911, LITTLEWOOD [7]) *If the complex sequence $\{na_n\}_{n=0}^{\infty}$ is bounded or, equivalently, if*

$$(1.1) \quad a_n = O\left(\frac{1}{n}\right), \quad (n \rightarrow \infty)$$

and if

$$(1.2) \quad f(x) := \sum_{n=0}^{\infty} a_n x^n \rightarrow A, \quad (x \uparrow 1)$$

then

$$(1.3) \quad \sum_{n=0}^{\infty} a_n = A.$$

Note the "subtle" difference with Tauber's theorem where we have the condition $a_n = o(1/n)$ instead of $a_n = O(1/n)$.

The above theorem is an easy consequence of the following (real)

THEOREM 1.2. (1914, HARDY & LITTLEWOOD [2]) *If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ with sum $f(x)$ and if $\lim_{x \uparrow 1} f(x) = A$, then the series $\sum_{n=0}^{\infty} a_n$ converges to A , provided (only) that there exists a constant G such that*

$$(1.4) \quad na_n \leq G \quad \text{for all } n.$$

Note the *one-sidedness* of condition (1.4) (in which the symbol \leq may just as well be replaced by \geq).

Since the proof of this theorem is not as simple as that of Tauber's first theorem (compare LANDAU [6; pp. 45-56]) we start by presenting a number of preparative lemmas.

LEMMA 1.1. (1930, KARAMATA [4]) *If $g(x)$ is defined for $0 \leq x < 1$ and if $\lim_{x \uparrow 1} (1-x)g(x) = A$, then, for every positive integer k ,*

$$(1.5) \quad \lim_{x \uparrow 1} (1-x)g(x^k) = \frac{A}{k} = A \int_0^1 t^{k-1} dt.$$

PROOF.

$$\lim_{x \uparrow 1} (1-x)g(x^k) = \lim_{x \uparrow 1} \frac{1-x}{1-x^k} (1-x^k)g(x^k) = \frac{1}{k} \cdot A = A \int_0^1 t^{k-1} dt. \quad \square$$

LEMMA 1.2. (1930, KARAMATA [4]) If $g(x) = \sum_{n=0}^{\infty} b_n x^n$ for $|x| < 1$ and if $\lim_{x \uparrow 1} (1-x)g(x) = A$, then

$$(1.6) \quad \lim_{x \uparrow 1} (1-x) \sum_{n=0}^{\infty} b_n x^n P(x^n) = A \int_0^1 P(t) dt$$

for every polynomial $P(t) = c_0 + c_1 t + \dots + c_m t^m$.

PROOF. Observe that

$$(1.7) \quad (1-x) \sum_{n=0}^{\infty} b_n x^n P(x^n) = (1-x) \sum_{n=0}^{\infty} b_n x^n \left(\sum_{k=0}^m c_k x^{kn} \right) = \\ = (1-x) \sum_{k=0}^m c_k \left\{ \sum_{n=0}^{\infty} b_n (x^{k+1})^n \right\} = \sum_{k=0}^m c_k (1-x)g(x^{k+1}).$$

Since by the previous lemma

$$(1.8) \quad \lim_{x \uparrow 1} (1-x)g(x^{k+1}) = A \int_0^1 t^k dt$$

the lemma follows. \square

LEMMA 1.3. (1930, KARAMATA [4]) If for $|x| < 1$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with all $b_n \geq 0$, and if $\lim_{x \uparrow 1} (1-x)g(x) = A$, then

$$(1.9) \quad \lim_{x \uparrow 1} (1-x) \sum_{n=0}^{\infty} b_n x^n \phi(x^n) = A \int_0^1 \phi(t) dt$$

for every real continuous function ϕ on the interval $[0,1]$.

PROOF. Since ϕ is continuous on $[0,1]$ there exists (for every $\varepsilon > 0$) a polynomial $\phi^* = \phi^*_\varepsilon$ such that

$$(1.10) \quad |\phi^*(x) - \phi(x)| \leq \frac{\varepsilon}{2}, \quad (0 \leq x \leq 1).$$

Defining the polynomials p and P by

$$(1.11) \quad p(x) = \phi^*(x) - \frac{\varepsilon}{2} \quad \text{and} \quad P(x) = \phi^*(x) + \frac{\varepsilon}{2}$$

we have

$$(1.12) \quad p(x) \leq \phi(x) \leq P(x), \quad (0 \leq x \leq 1).$$

Since all $b_n \geq 0$ we obtain

$$(1-x) \sum_{n=0}^{\infty} b_n x^n p(x^n) \leq (1-x) \sum_{n=0}^{\infty} b_n x^n \phi(x^n) \leq (1-x) \sum_{n=0}^{\infty} b_n x^n P(x^n).$$

Taking limits ($x \uparrow 1$) we find that

$$(1.13) \quad A \int_0^1 p(t) dt = \liminf_{x \uparrow 1} (1-x) \sum_{n=0}^{\infty} b_n x^n \phi(x^n) \leq \\ \leq \limsup_{x \uparrow 1} (1-x) \sum_{n=0}^{\infty} b_n x^n \phi(x^n) = A \int_0^1 P(t) dt.$$

Consequently

$$(1.14) \quad (\limsup_{x \uparrow 1} - \liminf_{x \uparrow 1}) (1-x) \sum_{n=0}^{\infty} b_n x^n \phi(x^n) \leq \\ \leq A \int_0^1 (P(t) - p(t)) dt = A\varepsilon.$$

Since $\varepsilon > 0$ may be chosen as small as we please it follows that

$$(1.15) \quad \lim_{x \uparrow 1} (1-x) \sum_{n=0}^{\infty} b_n x^n \phi(x^n) \text{ exists and } = L, \text{ say.}$$

Clearly

$$(1.16) \quad A \int_0^1 p(t) dt \leq L \leq A \int_0^1 P(t) dt$$

and, since $A \geq 0$ by the hypothesis that all $b_n \geq 0$, we also have

$$(1.17) \quad A \int_0^1 p(t) dt \leq A \int_0^1 \phi(t) dt \leq A \int_0^1 P(t) dt$$

so that

$$(1.18) \quad |L - A \int_0^1 \phi(t) dt| \leq A\varepsilon$$

from which it is clear that

$$(1.19) \quad L = A \int_0^1 \phi(t) dt$$

proving the lemma. \square

LEMMA 1.4. (1930, KARAMATA [4]) Lemma 1.3 also holds true if ϕ is Riemann-integrable over $[0,1]$ (instead of being continuous).

PROOF. Since ϕ is Riemann-integrable over $[0,1]$ we may construct two step-functions s and S on $[0,1]$ such that

$$(1.20) \quad s(x) \leq \phi(x) \leq S(x), \quad (0 \leq x \leq 1)$$

and

$$(1.21) \quad \int_0^1 (S(x) - s(x)) dx < \varepsilon.$$

By means of these step-functions we can construct two piecewise linear continuous functions ϕ_1 and ϕ_2 on $[0,1]$ such that

$$(1.22) \quad \phi_1(x) \leq \phi(x) \leq \phi_2(x), \quad (0 \leq x \leq 1)$$

and

$$(1.23) \quad \int_0^1 (\phi_2(x) - \phi_1(x)) dx \leq \varepsilon.$$

From here on the proof is similar to that of the previous lemma. \square

As an application we have

LEMMA 1.5. (1914, HARDY & LITTLEWOOD [2]) If for $|x| < 1$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$, with all $b_k \geq 0$, and if $\lim_{x \uparrow 1} (1-x)g(x) = A$, then

$$(1.24) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N b_n = A.$$

PROOF. Define the function ϕ as follows

$$(1.25) \quad \begin{cases} \phi(x) := 0 & \text{if } 0 \leq x < e^{-1} \\ \phi(x) := \frac{1}{x} & \text{if } e^{-1} \leq x \leq 1. \end{cases}$$

Then

$$(1.26) \quad \int_0^1 \phi(t) dt = \int_{e^{-1}}^1 \frac{1}{t} dt = -\log e^{-1} = 1.$$

By Lemma 1.4 we thus have

$$(1.27) \quad \lim_{x \uparrow 1} (1-x) \sum_{n=0}^{\infty} b_n x^n \phi(x^n) = A \int_0^1 \phi(t) dt = A.$$

Let $x = \exp(-\frac{1}{N})$ and observe that

$$(1.28) \quad \begin{aligned} (1-e^{-\frac{1}{N}}) \sum_{n=0}^{\infty} b_n e^{-\frac{n}{N}} \phi(e^{-\frac{n}{N}}) &= (1-e^{-\frac{1}{N}}) \sum_{n=0}^N b_n e^{-\frac{n}{N}} \phi(e^{-\frac{n}{N}}) = \\ &= (1-e^{-\frac{1}{N}}) \sum_{n=0}^N b_n = \frac{e^{-\frac{1}{N}} - 1}{-\frac{1}{N}} \frac{1}{N} \sum_{n=0}^N b_n. \end{aligned}$$

Combining this with (1.27) we obtain

$$(1.29) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{\infty} b_n = A. \quad \square$$

LEMMA 1.6. (1914, HARDY & LITTLEWOOD [2]) If the real function f is twice differentiable on $(0,1)$ such that $\lim_{x \uparrow 1} f(x) = A$ and

$$(1.30) \quad (1-x)^2 f''(x) \leq G, \quad (0 < x < 1)$$

for some constant G , then

$$(1.31) \quad \lim_{x \uparrow 1} (1-x) f'(x) = 0.$$

PROOF. We may assume that $G > 0$. Let $0 < \delta < 1$ and choose x_0 and x_1 such that $0 < x_0 < 1$ and $x_1 = x_0 + \delta(1-x_0)$, so that $x_0 < x_1 < 1$.

a. The Taylor expansion of f about x_0 reads

$$(1.32) \quad f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2} (x_1 - x_0)^2 f''(x_0 + \theta(x_1 - x_0))$$

for some θ between 0 and 1. This expansion may also be written as

$$(1.33) \quad f(x_1) = f(x_0) + \delta(1-x_0) f'(x_0) + \frac{1}{2} \delta^2 (1-x_0)^2 f''(x_0 + \theta \delta(1-x_0)).$$

From this we obtain

$$(1.34) \quad \begin{aligned} (1-x_0) f'(x_0) &= \frac{f(x_1) - f(x_0)}{\delta} - \frac{1}{2} \delta (1-x_0)^2 f''(x_0 + \theta \delta(1-x_0)) \geq \\ &\geq \frac{f(x_1) - f(x_0)}{\delta} - \frac{1}{2} \delta (1-x_0)^2 \frac{G}{(1 - (x_0 + \theta \delta(1-x_0)))^2} = \\ &= \frac{f(x_1) - f(x_0)}{\delta} - \frac{1}{2} \delta \frac{G}{(1-\theta\delta)^2} \geq \frac{f(x_1) - f(x_0)}{\delta} - \frac{1}{2} \frac{\delta G}{(1-\delta)^2} \end{aligned}$$

so that

$$(1.35) \quad \liminf_{x_0 \uparrow 1} (1-x_0) f'(x_0) \geq - \frac{\delta G}{2(1-\delta)^2}.$$

Since $\delta > 0$ may be chosen as small as we please it follows that

$$(1.36) \quad \liminf_{x \uparrow 1} (1-x) f'(x) \geq 0.$$

b. The Taylor expansion of f about x_1 reads

$$(1.37) \quad f(x_0) = f(x_1) + (x_0 - x_1) f'(x_1) + \frac{1}{2} (x_0 - x_1)^2 f''(x_1 + (1-\theta)(x_0 - x_1))$$

for some θ between 0 and 1. Since this may also be written as

$$(1.38) \quad \delta(1-x_0) f'(x_1) = f(x_1) - f(x_0) + \frac{1}{2} \delta^2 (1-x_0)^2 f''(x_1 + \theta \delta(1-x_0))$$

it follows that

$$\begin{aligned}
(1.39) \quad (1-x_1)f'(x_1) &= \frac{1-\delta}{\delta}(f(x_1)-f(x_0)) + \frac{1}{2}\delta(1-\delta)(1-x_0)^2 f''(x_0+\theta\delta(1-x_0)) \leq \\
&\leq \frac{1-\delta}{\delta}(f(x_1)-f(x_0)) + \frac{1}{2}\delta(1-\delta)(1-x_0)^2 \frac{G}{(1-(x_0+\theta\delta(1-x_0)))^2} = \\
&= \frac{1-\delta}{\delta}(f(x_1)-f(x_0)) + \frac{\delta(1-\delta)G}{2(1-\theta\delta)^2} \leq \\
&\leq \frac{1-\delta}{\delta}(f(x_1)-f(x_0)) + \frac{\delta G}{2(1-\delta)^2}
\end{aligned}$$

so that

$$(1.40) \quad \limsup_{x_1 \uparrow 1} (1-x_1)f'(x_1) \leq \frac{\delta G}{2(1-\delta)^2}$$

due to the fact that as $x_1 \uparrow 1$ then also $x_0 = \frac{x_1 - \delta}{1 - \delta} \uparrow 1$.
Since $\delta > 0$ may be chosen as small as we please we obtain

$$(1.41) \quad \limsup_{x \uparrow 1} (1-x)f'(x) \leq 0$$

and the lemma follows from a and b. \square

After these preparations we are ready for the

PROOF OF THEOREM 1.2. It is clear that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is twice differentiable on $(0,1)$ and that

$$(1.42) \quad f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}, \quad (0 < x < 1).$$

Since $na_n \leq G$ it follows that

$$(1.43) \quad (1-x)^2 f''(x) \leq (1-x)^2 G \sum_{n=2}^{\infty} (n-1)x^{n-2} = G.$$

By hypothesis we have $\lim_{x \uparrow 1} f(x) = A$ so that by Lemma 1.6

$$(1.44) \quad \lim_{x \uparrow 1} (1-x)f'(x) = 0$$

or

$$(1.45) \quad \lim_{x \uparrow 1} (1-x) \sum_{n=1}^{\infty} na_n x^{n-1} = 0.$$

Hence

$$(1.46) \quad \lim_{x \uparrow 1} (1-x) \sum_{n=1}^{\infty} \left(1 - \frac{na_n}{G}\right) x^{n-1} = \lim_{x \uparrow 1} (1-x) \sum_{n=0}^{\infty} \left(1 - \frac{(n+1)a_{n+1}}{G}\right) x^n = 1.$$

Since

$$(1.47) \quad 1 - \frac{na_n}{G} \geq 0 \quad \text{for all } n$$

it follows by Lemma 1.5 that

$$(1.48) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N+1} \left(1 - \frac{na_n}{G}\right) = 1$$

from which it is clear that

$$(1.49) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N na_n = 0.$$

Hence, invoking Tauber's second theorem, the proof is complete. \square

In 1952 Wielandt gave an interesting more direct proof of Theorem 1.1 avoiding the detour via Cesàro summability (Lemma 1.5). We present Korevaar's version of Wielandt's proof.

DIRECT PROOF OF THEOREM 1.1. (WIELANDT [9], KOREVAAR [5])

Define $a: [0, \infty) \rightarrow \mathbb{R}$ by

$$(1.50) \quad a(t) := a_n, \quad (n \leq t < n+1)$$

and note that

$$(1.51) \quad |a(t)| = \frac{2G + |a_0|}{t} \leq \frac{K}{1 - e^{-t}}, \quad (t > 0).$$

Also observe that, for $s > 0$,

$$(1.52) \quad \begin{aligned} \int_0^{\infty} a(t) e^{-st} dt &= \sum_{n=0}^{\infty} \int_n^{n+1} a(t) e^{-st} dt = \\ &= \sum_{n=0}^{\infty} a_n \frac{e^{-ns} - e^{-(n+1)s}}{s} = \frac{e^{-s} - 1}{-s} \sum_{n=0}^{\infty} a_n e^{-ns} = \frac{e^{-s} - 1}{-s} f(e^{-s}). \end{aligned}$$

Defining

$$(1.53) \quad g(x) := \begin{cases} 0 & \text{if } 0 \leq x < e^{-1} \\ 1 & \text{if } e^{-1} \leq x \leq 1 \end{cases}$$

we have

$$(1.54) \quad S_{n-1} := \sum_{k=0}^{n-1} a_k = \int_0^n a(t) dt = n \int_0^1 a(nt) dt = n \int_0^{\infty} a(nt) g(e^{-t}) dt.$$

Now let $\varepsilon > 0$ be given and determine a polynomial $p(x)$ such that

$$(1.55) \quad \int_0^1 \left| \frac{g(x)-x}{x(1-x)} - p(x) \right| dx < \varepsilon.$$

Then we have

$$(1.56) \quad \int_0^\infty \frac{|g(e^{-v}) - P(e^{-v})|}{1-e^{-v}} dv < \varepsilon$$

where $P(x) := x + x(1-x)p(x) = \sum_{k=1}^m c_k x^k$, so that $P(0) = 0$.

It follows that

$$(1.57) \quad \begin{aligned} |S_{n-1} - \sum_{k=1}^m c_k n \int_0^\infty a(nt) e^{-kt} dt| &= \\ &= n \left| \int_0^\infty a(nt) (g(e^{-t}) - P(e^{-t})) dt \right| \leq \\ &\leq n \int_0^\infty \frac{K}{nt} |g(e^{-t}) - P(e^{-t})| dt \leq \\ &\leq K \int_0^\infty \frac{|g(e^{-t}) - P(e^{-t})|}{1-e^{-t}} dt \leq K\varepsilon \end{aligned}$$

so that

$$(1.58) \quad S_{n-1} = \sum_{k=1}^m c_k n \int_0^\infty a(nt) e^{-kt} dt + \theta K\varepsilon = \sum_{k=1}^m c_k \int_0^\infty a(t) e^{-\frac{kt}{n}} dt + \theta K\varepsilon$$

for some θ with $|\theta| \leq 1$. This result may also be written as

$$(1.59) \quad S_{n-1} = \sum_{k=1}^m c_k \frac{e^{-\frac{k}{n}} - 1}{-\frac{k}{n}} f(e^{-\frac{k}{n}}) + \theta K\varepsilon.$$

Without loss of generality we may assume that $\lim_{x \uparrow 1} f(x) = A = 0$ so that

$$(1.60) \quad \limsup_{n \rightarrow \infty} |S_n| \leq |\theta| K\varepsilon.$$

Since $\varepsilon > 0$ may be chosen as small as we please it follows that

$$(1.61) \quad \lim_{n \rightarrow \infty} S_n = 0. \quad \square$$

The integral analogue of Theorem 1.2 reads

THEOREM 1.3. If $\lim_{s \downarrow 0} \int_0^{\infty} e^{-st} F(t) dt = A$ and if there exists a constant G such that $F(t) \leq \frac{G}{t}$ for all $t > 0$, then $\int_0^{\infty} F(t) dt = A$.

PROOF. See DOETSCH [1; p. 516] or WIDDER [8; pp. 195-196].

A simple application of the last theorem is the following.

Let

$$(1.62) \quad F(t) := \frac{\sin t}{t}, \quad (t > 0)$$

and

$$(1.63) \quad \phi(s) := \int_0^{\infty} e^{-st} F(t) dt, \quad (s > 0).$$

Then $\phi'(s) = -(s^2+1)^{-1}$, from which it is easily seen that

$$(1.64) \quad \phi(s) = \frac{\pi}{2} - \arctan(s).$$

Since $F(t) \leq \frac{1}{t}$ for $t > 0$, it follows that

$$(1.65) \quad \int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

2. SOME EXTENSIONS TO GENERAL DIRICHLET-SERIES

A general Dirichlet-series (D-series) is a series of the form

$$(2.1) \quad \sum_{n=0}^{\infty} a_n e^{-s\lambda_n} \quad (=: D(s), \text{ if the series converges})$$

with s and all a_n complex, and $\lambda_0 < \lambda_1 < \lambda_2 < \dots \rightarrow \infty$.

Quite frequently it is assumed that $\lambda_0 = 0$.

Note that a D-series is a generalization of a power series $\sum_{n=0}^{\infty} a_n x^n$ by taking $x = e^{-s}$, and also of a special D-series $\sum_{n=1}^{\infty} a_n n^{-s}$ by taking $\lambda_n = \log(n+1)$.

We conclude this chapter by listing some Tauberian theorems for D-series. For the proofs we refer to the literature, in particular HARDY & RIESZ [3] (and HARDY's *Divergent Series*).

THEOREM 2.0. (Abel) If $\sum_{n=0}^{\infty} a_n = A$ then $\lim_{s \downarrow 0} D(s) = A$.

THEOREM 2.1. (Tauber) If $\lim_{s \downarrow 0} D(s) = A$ and $a_n = o\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$

as $n \rightarrow \infty$, then $\sum_{n=0}^{\infty} a_n = A$.

THEOREM 2.2 (Tauber) If $\lim_{s \downarrow 0} D(s) = A$ and $\sum_{k=1}^n \lambda_k a_k = o(\lambda_n)$

as $n \rightarrow \infty$, then $\sum_{n=0}^{\infty} a_n = A$.

THEOREM 2.3. (Hardy) If $\lim_{s \rightarrow 0} sD(s) = L$ and all $a_n \geq 0$, then

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k=0}^n a_k = L.$$

THEOREM 2.4. (Littlewood) If $\lim_{s \rightarrow 0} D(s) = A$ and $a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$

as $n \rightarrow \infty$, then $\sum_{n=0}^{\infty} a_n = A$.

THEOREM 2.5. (Hardy & Littlewood) If $\lim_{s \rightarrow 0} D(s) = A$ and if there exists a constant G such that $a_n \leq G \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}$ for all $n \geq 1$, then $\sum_{n=0}^{\infty} a_n = A$.

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CHAPTER 3

SOME APPLICATIONS

1. THE SERIES $\sum_{n=1}^{\infty} n^{-1-it}$ IS NOT ABEL-SUMMABLE FOR $t \in \mathbb{R}$

If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ and if $\lim_{x \uparrow 1} \sum_{n=0}^{\infty} a_n x^n = S$, then the series $\sum_{n=0}^{\infty} a_n$ is called *Abel-summable* to the (Abel-)sum S . Notation: (A) $\sum_{n=0}^{\infty} a_n = S$.

Using this terminology we may express, for example, Theorem 1.1 in Chapter 2 as follows: If (A) $\sum_{n=0}^{\infty} a_n = S$ and $a_n = O(\frac{1}{n})$, then $\sum_{n=0}^{\infty} a_n = S$.

As an application of this theorem we will show that the series $\sum_{n=1}^{\infty} n^{-1-it}$ is *not* Abel-summable for any $t \in \mathbb{R}$.

For if it were, it would follow (since $|n^{-1-it}| = \frac{1}{n}$) that $\sum_{n=1}^{\infty} n^{-1-it}$ is convergent. However, this is not the case as we shall show below.

For $t = 0$ the situation is clear: $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

For $t \neq 0$ we may argue as follows

$$\begin{aligned}
 (1.1) \quad \sum_{n=1}^N n^{-1-it} &= \int_{1-0}^N x^{-1-it} d[x] = x^{-1-it} [x] \Big|_{1-0}^N + \\
 &- \int_1^N [x] dx^{-1-it} = N^{-it} - \int_1^N [x] dx^{-1-it} = \\
 &= N^{-it} + (1+it) \int_1^N \frac{[x]-x}{x^{2+it}} dx + (1+it) \int_1^N \frac{dx}{x^{1+it}} = \\
 &= -\frac{N^{-it}}{it} + \frac{1+it}{it} + (1+it) \int_1^N \frac{[x]-x}{x^{2+it}} dx.
 \end{aligned}$$

Since $\int_1^{\infty} \frac{[x]-x}{x^{2+it}} dx$ is an absolutely convergent integral, it follows

that for some constant C

$$(1.2) \quad \sum_{n=1}^N n^{-1-it} = -\frac{N^{-it}}{it} + C + o(1), \quad (N \rightarrow \infty)$$

from which it is clear that $\sum_{n=1}^{\infty} n^{-1-it}$ diverges. \square

2. AN ALTERNATIVE PROOF OF A THEOREM OF JORDAN

Let the function $f : \mathbb{R} \rightarrow \mathbb{C}$ be periodic with period 2π and let f be integrable (in the sense of Lebesgue) over $[0, 2\pi]$. The Fourier coefficients of f are defined by

$$(2.1) \quad c_n := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx, \quad (n \in \mathbb{Z})$$

and the series

$$(2.2) \quad \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad (x \in \mathbb{R})$$

is called the Fourier series of f .

Such a series is called convergent if the series

$$(2.3) \quad c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx})$$

is convergent. (Even if this series converges its sum need not be equal to $f(x)$.)

As to the convergence of such a series we have the following theorem (Jordan): *If f is of bounded variation in the neighbourhood of the point $x \in \mathbb{R}$, then the Fourier series of f converges to (see TITCHMARSH [4; pp. 406-407])*

$$(2.4) \quad \frac{1}{2}(f(x+0) + f(x-0)).$$

Below we will prove this theorem by means of Theorem 1.1 of Chapter 2.

DEFINITION. The series $\sum_{n=0}^{\infty} a_n$ is called *Cesàro-summable* to the (Cesàro-)sum S if

$$(2.5) \quad \lim_{n \rightarrow \infty} \frac{1}{n}(S_0 + S_1 + \dots + S_n) = S$$

where $S_k := a_0 + a_1 + \dots + a_k$. Notation: (C) $\sum_{n=0}^{\infty} a_n = S$.

From the theory of Fourier series we borrow the following theorem (Fejér): *The Fourier series of f is Cesàro-summable to $\frac{1}{2}(f(x+0) + f(x-0))$ for every value of x for which this expression exists (see TITCHMARSH [4; p. 414]).*

Furthermore, it can be shown that if a series is Cesàro-summable to the C-sum S , then it is also Abel-summable to the A-sum S (see HARDY [1; p. 108] or KNOPP [3; p. 508]). In other words, Abel-summation is stronger than Cesàro-summation.

Combining Theorem 1.1 of Chapter 2 and Fejér's theorem, Jordan's theorem will follow if we can show that $c_n = O\left(\frac{1}{|n|}\right)$ as $n \rightarrow \infty$. For a proof of this fact for functions of bounded variation we refer to TITCHMARSH [4; pp. 426-427]. \square

REMARK. In our approach to Jordan's theorem it is the property $c_n = O\left(\frac{1}{|n|}\right)$ that does the work. For more on the order of magnitude of Fourier coefficients we refer to the standard treatises on Fourier series, for example, HARDY & ROGOSINSKI [2] and ZYGMUND [5].

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CHAPTER 4

INTRODUCTION TO PITT'S GENERAL TAUBERIAN THEOREM

For some mathematicians the rather technical proofs of the previous Tauberian theorems are interesting and impressive because they make (quoting Wiener) appreciable demands on analytical technique. Others feel that these theorems and their proofs do not quite satisfy their demands on transparency and insight. N. Wiener was the first to remove this deficiency by creating a theory from which, for example, Littlewood's theorem is a rather simple consequence.

We will not immediately begin with Wiener's (general Tauberian) theory but work our way up from Littlewood's theorem to the more general ideas of Wiener. We prefer to start with the introduction of Pitt's general Tauberian theorem for slowly oscillating functions.

The hypotheses in Littlewood's theorem are

$$(a) \quad \exists G : |na_n| \leq G, \quad (n \geq 1)$$

$$(b) \quad \lim_{x \uparrow 1} f(x) = \lim_{x \uparrow 1} \sum_{n=0}^{\infty} a_n x^n = A.$$

Without loss of generality we may assume that $A = 0$. From the above conditions we will derive a number of consequences which, later on, will serve as hypotheses in a more general theorem.

LEMMA 1. If $0 < x \leq y$, then

$$(1) \quad \sum_{x \leq n \leq y} \frac{1}{n} \leq \log \frac{y}{x} + \min(1, \frac{1}{x}).$$

PROOF. In case the interval $[x, y]$ contains no integers the lemma is clearly true. If $[x, y]$ contains the integers n_0, n_0+1, \dots, n_0+m ($m \geq 0$), then

$$(2) \quad \sum_{x \leq n \leq y} \frac{1}{n} = \frac{1}{n_0} + \sum_{k=1}^m \frac{1}{n_0+k} \leq \min(1, \frac{1}{x}) + \int_x^y \frac{1}{t} dt. \quad \square$$

As an immediate consequence we obtain

$$\text{LEMMA 2. If } x > 0 \text{ and } \rho := \frac{y}{x} \geq 1, \text{ then } \sum_{x \leq n \leq y} \frac{1}{n} \leq 1 + \log \rho.$$

Now define

$$(3) \quad S(v) := \sum_{n < v} a_n, \quad (v \in \mathbb{R})$$

and note that $S(v)$ is continuous from the left.

LEMMA 3. $S(v) = O(\log v)$, $(v \rightarrow \infty)$

and

$$(4) \quad f(e^{-\frac{1}{u}}) = \int_0^{\infty} S(uv)e^{-v} dv, \quad (u > 0).$$

PROOF. For $v \geq 1$ we have

$$(5) \quad |S(v)| \leq \sum_{n < v} |a_n| \leq |a_0| + \sum_{1 \leq n < v} \frac{n|a_n|}{n} \leq \\ \leq |a_0| + G \sum_{1 \leq n \leq v} \frac{1}{n} \leq |a_0| + G(1 + \log v)$$

proving the first assertion.

For $u > 0$ we have

$$(6) \quad f(e^{-\frac{1}{u}}) = \sum_{n=0}^{\infty} a_n e^{-\frac{n}{u}} = \int_0^{\infty} e^{-\frac{t}{u}} dS(t) = \\ = e^{-\frac{t}{u}} S(t) \Big|_{t=0}^{t=\infty} + \frac{1}{u} \int_0^{\infty} S(t) e^{-\frac{t}{u}} dt = \int_0^{\infty} S(uv) e^{-v} dv$$

proving the second assertion. \square

DEFINITION. A function $\phi : (x_0, \infty) \rightarrow \mathbb{C}$ is called *slowly oscillating* if

$$(7) \quad \forall \varepsilon > 0 \exists \delta_\varepsilon, N_\varepsilon : |\phi(\rho x) - \phi(x)| < \varepsilon$$

for all ρ, x satisfying $1 \leq \rho < 1 + \delta_\varepsilon$ and $x \geq N_\varepsilon$.

EXAMPLE: $\log x$ is slowly oscillating on $(0, \infty)$.

LEMMA 4. $S(x)$ is slowly oscillating.

PROOF. Let $\varepsilon \in (0, 1)$ be given. Then, for $\rho \geq 1$ and $x > 0$,

$$(8) \quad |S(\rho x) - S(x)| \leq \sum_{x \leq n < \rho x} |a_n| \leq \sum_{x \leq n \leq \rho x} \frac{|na_n|}{n} \leq \\ \leq G \sum_{x \leq n \leq \rho x} \frac{1}{n} \leq G(\log \rho + \min(1, \frac{1}{x})).$$

Now choose $\delta_\varepsilon = -1 + \exp(\frac{\varepsilon}{2G})$ and $N_\varepsilon = \frac{2G}{\varepsilon}$.

Then $|S(\rho x) - S(x)| < \epsilon$ for all ρ, x satisfying $1 \leq \rho < 1 + \delta_\epsilon$ and $x \geq N_\epsilon$, proving that $S(x)$ is slowly oscillating. \square

LEMMA 5. $S(x)$ is bounded.

PROOF.

$$\begin{aligned}
 (9) \quad |f(e^{-\frac{1}{x}}) - S(x)| &= \left| \int_0^\infty S(xv) e^{-v} dv - \int_0^\infty S(x) e^{-v} dv \right| \leq \\
 &\leq \int_0^\infty e^{-v} |S(xv) - S(x)| dv = \\
 &= \int_0^1 e^{-v} \left| \sum_{vx \leq n < x} a_n \right| dv + \int_1^\infty e^{-v} \left| \sum_{x \leq n < vx} a_n \right| dv \leq \\
 &\leq G \int_0^\infty (1 + |\log v|) e^{-v} dv =: K.
 \end{aligned}$$

Hence $|S(x)| \leq K + |f(e^{-\frac{1}{x}})|$ and since $\lim_{x \rightarrow \infty} f(e^{-\frac{1}{x}})$ exists it follows that $S(x)$ is bounded. \square

After these preparations the reader will have no difficulty to see that Littlewood's theorem is a straightforward consequence of the following (general) Tauberian

THEOREM 1. (Pitt) If the measurable function $S(v)$ is bounded and slowly oscillating on $(0, \infty)$ and if $\int_0^\infty e^{-v} S(uv) dv \rightarrow 0$ as $u \rightarrow \infty$, then $\lim_{v \rightarrow \infty} S(v) = 0$.

The proof of this theorem will be given in Chapter 11.

One may try to generalize this theorem by replacing the kernel e^{-v} by some other kernel $k(v) \in L^1(0, \infty)$. The result would read:

If $S(v)$ is bounded and slowly oscillating on $(0, \infty)$ and if $\lim_{u \rightarrow \infty} \int_0^\infty k(v) S(uv) dv = 0$ for some $k \in L^1(0, \infty)$, then $\lim_{v \rightarrow \infty} S(v) = 0$.

In order to investigate the question whether this is a true theorem indeed we suppose that there exists an $x_0 \in \mathbb{R}$ such that $\int_0^\infty v^{x_0} k(v) dv = 0$. Observing that the function $\sigma(v) := v^{x_0}$ is bounded and slowly oscillating and that

$$(10) \quad \int_0^\infty \sigma(uv) k(v) dv = \int_0^\infty (uv)^{x_0} k(v) dv = u^{x_0} \int_0^\infty v^{x_0} k(v) dv \equiv 0$$

we certainly have

$$(11) \quad \lim_{u \rightarrow \infty} \int_0^\infty \sigma(uv) k(v) dv = 0.$$

However, $\sigma(v) = v^{x_0 i}$ does not tend to zero as $v \rightarrow \infty$. Hence, in order to obtain a *correct* theorem we certainly have to stipulate that $\int_0^\infty v^{xi} k(v) dv \neq 0$ for all $x \in \mathbf{R}$. It was shown by Pitt that we get the following

THEOREM 2. (Pitt) If $S(v)$ is bounded and slowly oscillating on $(0, \infty)$ and if $\lim_{u \rightarrow \infty} \int_0^\infty k(v) S(uv) dv = 0$ for some $k \in L^1(0, \infty)$ satisfying

$$(12) \quad \int_0^\infty v^{xi} k(v) dv \neq 0 \quad \text{for all } x \in \mathbf{R}$$

then

$$(13) \quad \lim_{v \rightarrow \infty} S(v) = 0.$$

In accordance with the introductory character of this chapter we will not prove this theorem here but defer its proof to Chapter 11.

It should be remarked that, from the historical point of view, Pitt's theorem was discovered after Wiener had built up his general Tauberian theory which will be introduced in the next chapter.

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INTRODUCTION TO WIENER'S GENERAL TAUBERIAN THEOREM

1. FIRST APPROACH

In Chapter 3 we already mentioned that Cesàro-summability implies Abel-summability. The converse, however, is not generally true as may be seen from an example as described in Knopp's *Theorie und Anwendung der unendlichen Reihen*, pp. 516-517.

If we impose some (Tauberian) condition on the terms of the series in question, then the statement "A-summable \Rightarrow C-summable" may be true. In this vein we have the following Tauberian

THEOREM 1. If $\sum_{n=0}^{\infty} a_n$ is Abel-summable to the sum S and if $S_n := a_0 + a_1 + \dots + a_n$ is bounded, then $\sum_{n=0}^{\infty} a_n$ is Cesàro-summable to S .

PROOF. Without loss of generality we may assume that $a_0 = 0$ and $S = 0$. As before we let $w_0 = 0$ and $w_n = a_1 + 2a_2 + \dots + na_n$ so that

$$(1.1) \quad \begin{aligned} w_n &= (S_1 - S_0) + 2(S_2 - S_1) + \dots + n(S_n - S_{n-1}) = \\ &= (n+1)S_n - (S_1 + S_2 + \dots + S_n) \end{aligned}$$

from which it follows that $w_n = o(n)$ as $n \rightarrow \infty$.

Defining $v_n := \frac{w_n}{n(n+1)}$, we thus have $v_n = o(\frac{1}{n})$ as $n \rightarrow \infty$.

Writing

$$(1.2) \quad f(x) = \sum_{n=1}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} v_n x^{n+1}, \quad (|x| < 1)$$

we have

$$(1.3) \quad \begin{aligned} g(x) + (1-x)g'(x) &= \\ &= \sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} x^{n+1} + \sum_{n=1}^{\infty} \frac{w_n}{n} x^n - \sum_{n=1}^{\infty} \frac{w_n}{n} x^{n+1} = \\ &= - \sum_{n=1}^{\infty} \frac{w_n}{n+1} x^{n+1} + \sum_{n=1}^{\infty} \frac{w_n}{n} x^n = \sum_{n=1}^{\infty} \frac{w_n - w_{n-1}}{n} x^n = f(x) \end{aligned}$$

so that, since $S = 0$,

$$(1.4) \quad g(x) + (1-x)g'(x) = o(1), \quad (x \uparrow 1)$$

or, equivalently,

$$(1.5) \quad \frac{d}{dx} \frac{g(x)}{1-x} = \frac{o(1)}{(1-x)^2}, \quad (x \uparrow 1).$$

Integrating this result over $[0, t]$ with $0 < t < 1$ we obtain

$$(1.6) \quad \frac{g(t)}{1-t} = o\left(\frac{1}{1-t}\right), \quad (t \uparrow 1)$$

or

$$(1.7) \quad g(t) = o(1), \quad (t \uparrow 1).$$

Since $v_n = O\left(\frac{1}{n}\right)$ it follows from Littlewood's theorem that

$$(1.8) \quad \sum_{n=1}^{\infty} v_n = 0.$$

Observing that (see (1.1))

$$(1.9) \quad \begin{aligned} \sum_{n=1}^N v_n &= \sum_{n=1}^N w_n \left(\frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^N \frac{w_n - w_{n-1}}{n} - \frac{w_N}{N+1} = \\ &= \sum_{n=1}^N a_n - \frac{w_N}{N+1} = \frac{S_1 + S_2 + \dots + S_N}{N+1} \end{aligned}$$

it follows that

$$(1.10) \quad \lim_{N \rightarrow \infty} \frac{S_1 + S_2 + \dots + S_N}{N+1} = \sum_{n=1}^{\infty} v_n = 0.$$

Hence $\sum_{n=1}^{\infty} a_n$ is Cesàro-summable to the sum 0, completing the proof. \square

In the above theorem the condition (A) $\sum_{n=0}^{\infty} a_n = S$ may also be expressed as

$$(1.11) \quad (1-r) \sum_{n=0}^{\infty} S_n r^n \rightarrow S, \quad (r \uparrow 1)$$

or

$$(1.12) \quad (1-e^{-x}) \sum_{n=0}^{\infty} S_n e^{-\frac{n}{x}} \rightarrow S, \quad (x \rightarrow \infty)$$

or

$$(1.13) \quad \frac{1}{x} \sum_{n=0}^{\infty} S_n e^{-\frac{n}{x}} \rightarrow S, \quad (x \rightarrow \infty).$$

Hence, Theorem 1 may also be formulated as

THEOREM 2. If $\frac{1}{x} \sum_{n=0}^{\infty} S_n e^{-\frac{n}{x}} \rightarrow S$, $(x \rightarrow \infty)$, and if S_n is bounded (i.e. $S_n = O(1)$), then $\frac{1}{N} \sum_{n=0}^N S_n \rightarrow S$, $(N \rightarrow \infty)$.

The integral version of this theorem reads

THEOREM 3. If $F(x)$ is bounded and measurable on $(0, \infty)$ and

$$(1.14) \quad \frac{1}{x} \int_0^{\infty} e^{-\frac{t}{x}} F(t) dt \rightarrow L, \quad (x \rightarrow \infty)$$

then

$$(1.15) \quad \frac{1}{x} \int_0^x F(t) dt \rightarrow L, \quad (x \rightarrow \infty).$$

We will not prove this theorem here but use it as an introductory means for Wiener's theorem, of which it is also a consequence.

We reformulate Theorem 3 by means of the following functions

$$(1.16) \quad G_1(t) = e^{-t} \quad \text{and} \quad G_2(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}.$$

Now the *hypothesis* in Theorem 3 may be written as

$$(h) \quad \frac{1}{x} \int_0^{\infty} G_1\left(\frac{t}{x}\right) F(t) dt \rightarrow L \int_0^{\infty} G_1(t) dt, \quad (x \rightarrow \infty)$$

and its *conclusion* as

$$(c) \quad \frac{1}{x} \int_0^{\infty} G_2\left(\frac{t}{x}\right) F(t) dt \rightarrow L \int_0^{\infty} G_2(t) dt, \quad (x \rightarrow \infty).$$

By means of the transformations

$$(1.17) \quad x = e^{\xi}; \quad t = e^{\tau}; \quad F(e^{\tau}) = f(\tau); \quad e^{\tau} G_j(e^{\tau}) = g_j(-\tau), \quad (j=1,2)$$

Theorem 3 may thus be put into the following form (after having replaced ξ by x , and τ by t).

THEOREM 4. If

$$(1.18) \quad \int_{-\infty}^{\infty} g_1(x-t) f(t) dt \rightarrow L \int_{-\infty}^{\infty} g_1(t) dt, \quad (x \rightarrow \infty)$$

and if f is bounded on \mathbf{R} , then

$$(1.19) \quad \int_{-\infty}^{\infty} g_2(x-t) f(t) dt \rightarrow L \int_{-\infty}^{\infty} g_2(t) dt, \quad (x \rightarrow \infty).$$

We are thus led to ask the question under which conditions on g_1 and g_2 Theorem 4 still holds true. Wiener discovered that there is a general condition on g_1 which almost enables us to dispense with any condition on g_2 . Wiener's condition is that the Fourier transform of g_1 does not vanish on \mathbf{R} .

This condition is suggested by the theory of Fourier transforms of functions of the class $L^1(\mathbb{R})$. In order to illustrate this we write $P(g,f)$ for the property

$$(1.20) \quad \int_{-\infty}^{\infty} g(x-t) f(t) dt \rightarrow L \int_{-\infty}^{\infty} g(t) dt, \quad (x \rightarrow \infty)$$

where L is some complex constant.

Assuming that $P(g,f)$ holds true and defining $h : \mathbb{R} \rightarrow \mathbb{C}$ by

$$(1.21) \quad h(x) = \sum_{m=1}^n r_m g(x-a_m), \quad (r_m \in \mathbb{C}; a_m \in \mathbb{R})$$

we see that $P(h,f)$ also holds true. This suggests that, with proper precautions, that $P(h,f)$ also holds true in case h is defined by

$$(1.22) \quad h(x) := \int_{-\infty}^{\infty} r(u) g(x-u) du.$$

We are thus led to ask whether, given $g \in L^1(\mathbb{R})$, an arbitrary $h \in L^1(\mathbb{R})$ can be represented as $h = r * g$ where $*$ denotes the *convolution product*.

For $r \in L^1(\mathbb{R})$ the Fourier transform \hat{r} is defined by

$$(1.23) \quad \hat{r}(t) := \int_{-\infty}^{\infty} r(u) e^{-itu} du.$$

By a well known property of Fourier transforms it is thus required that $\hat{h}(t) = \hat{r}(t) \cdot \hat{g}(t)$ for all $t \in \mathbb{R}$. Hence, if we want to express h as $r * g$ for some r , then we are led to take \hat{r} such that $\hat{r}(t) = \hat{h}(t)/\hat{g}(t)$ and r itself (in some sense, by the inversion formula for Fourier transforms) such that

$$(1.24) \quad r(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{r}(t) e^{itx} dt.$$

It seems that this *formal* procedure can only be successful if $\hat{g}(t) \neq 0$ for all $t \in \mathbb{R}$.

We define $W(\mathbb{R})$ as the class of all functions belonging to $L^1(\mathbb{R})$ whose Fourier transforms do *not vanish* on \mathbb{R} . If $g \in W(\mathbb{R})$ and $h \in L^1(\mathbb{R})$ are given, the equation $h = r * g$ may still not be solvable, but we can solve this equation approximately (details will be given later), and this will be sufficient to prove

(Wiener's general Tauberian) THEOREM. If $g_1 \in W(\mathbb{R})$ and $f \in BM(\mathbb{R})$ (i.e. f is bounded and measurable), then

$$(1.25) \quad P(g_1, f) \Rightarrow P(g_2, f)$$

for all $g_2 \in L^1(\mathbb{R})$.

We will return to this subject in Chapter 8.

2. SECOND APPROACH

Another way of looking at Wiener's theorem is the following.

From the assumptions $f \in BM(\mathbb{R})$, $g \in L^1(\mathbb{R})$ and

$$(2.1) \quad \int_{-\infty}^{\infty} g(x-t) f(t) dt \rightarrow L \int_{-\infty}^{\infty} g(t) dt, \quad (x \rightarrow \infty)$$

we want to conclude (under suitable conditions) that

$$(2.2) \quad \int_{-\infty}^{\infty} g^*(x-t) f(t) dt \rightarrow L \int_{-\infty}^{\infty} g^*(t) dt, \quad (x \rightarrow \infty)$$

for every $g^* \in L^1(\mathbb{R})$.

Let, for any function $g : \mathbb{R} \rightarrow \mathbb{C}$, the translate g_a be defined by

$$(2.3) \quad g_a(t) := g(t+a), \quad (t \in \mathbb{R})$$

where a is some real number.

Clearly (2.2) holds true if g^* is a finite linear combination of translates of g , i.e. for any h of the form

$$(2.4) \quad h := \sum_{n=1}^N c_n g_{a_n}, \quad (c_n \in \mathbb{C}; a_n \in \mathbb{R}; g \in L^1(\mathbb{R})).$$

Let T_g be the set of all these functions h . It is easy to see that (2.2) also holds true for all functions in $L^1(\mathbb{R})$ which can be approximated (in the norm of $L^1(\mathbb{R})$) arbitrarily close by functions from T_g . Hence, the question arises under what conditions one has that the set T_g is *dense* in $L^1(\mathbb{R})$. In other words: when is the closure \overline{T}_g of T_g equal to the whole space $L^1(\mathbb{R})$? The answer is

$$(2.5) \quad \overline{T}_g = L^1(\mathbb{R}) \Leftrightarrow g \in W(\mathbb{R}).$$

In Chapter 8 we will see that this statement is equivalent to Wiener's general Tauberian theorem described in Section 1.

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CHAPTER 6

FOURIER TRANSFORMS

1. THE SIMPLEST PROPERTIES OF FOURIER TRANSFORMS

As usual we denote by $L^1(\mathbb{R})$, or briefly L^1 , the set of all Lebesgue measurable functions $f : \mathbb{R} \rightarrow \mathbb{C}$ for which $\int_{-\infty}^{\infty} |f(x)| dx$ is finite. For $f \in L^1$ we define the *Fourier transform* \hat{f} of f by

$$(1.1) \quad \hat{f}(x) := \int_{-\infty}^{\infty} e^{ixt} f(t) dt, \quad (x \in \mathbb{R}).$$

PROPOSITION 1.1. \hat{f} is bounded on \mathbb{R} for every $f \in L^1$.

$$\begin{aligned} \text{PROOF. } |\hat{f}(x)| &= \left| \int_{-\infty}^{\infty} e^{ixt} f(t) dt \right| \leq \int_{-\infty}^{\infty} |e^{ixt} f(t)| dt = \\ &= \int_{-\infty}^{\infty} |f(t)| dt < \infty. \end{aligned}$$

Hence

$$\sup_{x \in \mathbb{R}} |\hat{f}(x)| \leq \int_{-\infty}^{\infty} |f(t)| dt$$

a result that is usually written as

$$(1.2) \quad \|\hat{f}\|_{\infty} \leq \|f\|_1.$$

PROPOSITION 1.2. $\hat{f} \in C(\mathbb{R})$ for every $f \in L^1$. ($C(\mathbb{R})$ denotes the set of all continuous complex functions on \mathbb{R} .)

Before proving this proposition we recall

LEBESGUE'S DOMINATED CONVERGENCE THEOREM (LDCT, for short):

If $f_n \in L^1$ for all $n \in \mathbb{N}$ and if there exists an $F \in L^1$ such that

$$|f_n(x)| \leq F(x) \quad \text{a.e. on } \mathbb{R}$$

and if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{a.e. on } \mathbb{R}$$

then $f \in L^1$ and

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} f(x) dx.$$

PROOF OF PROPOSITION 1.2. For real x and h we have

$$(1.3) \quad \hat{f}(x+h) - \hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} (e^{iht} - 1) f(t) dt$$

so that

$$(1.4) \quad |\hat{f}(x+h) - \hat{f}(x)| \leq \int_{-\infty}^{\infty} |e^{iht} - 1| |f(t)| dt.$$

Now observe that

$$(1.5) \quad |e^{iht} - 1| |f(t)| \leq 2|f(t)|, \quad 2|f| \in L^1$$

and that for every real sequence $\{h_n\}_{n=1}^{\infty}$ with $\lim_{n \rightarrow \infty} h_n = 0$

$$(1.6) \quad \lim_{n \rightarrow \infty} |e^{ih_n t} - 1| |f(t)| = 0, \quad \text{a.e.}$$

so that, by LDCT,

$$(1.7) \quad \lim_{h \rightarrow 0} \int_{-\infty}^{\infty} |e^{iht} - 1| |f(t)| dt = \int_{-\infty}^{\infty} 0 dt = 0.$$

It follows that \hat{f} is (uniformly) continuous on \mathbb{R} . \square

PROPOSITION 1.3. $\hat{f} \in C_0(\mathbb{R})$ for every $f \in L^1$. ($C_0(\mathbb{R})$ denotes the set of all continuous functions ψ on \mathbb{R} which vanish at infinity, i.e. for which $\lim_{|x| \rightarrow \infty} \psi(x) = 0$.)

Before proving this proposition we recall that if $f \in L^1$ then

$$(1.8) \quad \lim_{t \rightarrow 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)| dx = 0.$$

This property may also be formulated as follows. For $a \in \mathbb{R}$ and $f \in L^1$ let the a -translate f_a of f be defined as before (p. 32):

$$(1.9) \quad f_a(x) = f(x+a), \quad (x \in \mathbb{R})$$

and let the map $\phi_f : \mathbb{R} \rightarrow L^1$ be defined by

$$(1.10) \quad \phi_f(a) = f_a, \quad (a \in \mathbb{R}; f \text{ (fixed)} \in L^1).$$

Property (1.8) may then be stated as

$$(1.11) \quad \lim_{t \rightarrow 0} \|f_t - f\|_1 = \lim_{t \rightarrow 0} \|\phi_f(t) - \phi_f(0)\|_1 = 0$$

which is equivalent to saying that the map ϕ_f is continuous at $t = 0$. (Actually, ϕ_f is continuous on all of \mathbb{R} .)

PROOF OF PROPOSITION 1.3. We only need to show that $\lim_{|x| \rightarrow \infty} \hat{f}(x) = 0$.
By definition we have

$$(1.12) \quad \hat{f}(x) = \int_{-\infty}^{\infty} e^{itx} f(t) dt$$

so that, for $x \neq 0$,

$$(1.13) \quad -\hat{f}(x) = e^{\pi i} \hat{f}(x) = \int_{-\infty}^{\infty} e^{ix(t+\frac{\pi}{x})} f(t) dt = \int_{-\infty}^{\infty} e^{ixu} f(u-\frac{\pi}{x}) du.$$

From this it follows that

$$(1.14) \quad 2\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} \{f(t) - f(t-\frac{\pi}{x})\} dt$$

so that

$$(1.15) \quad 2|\hat{f}(x)| \leq \int_{-\infty}^{\infty} |f(t) - f(t-\frac{\pi}{x})| dt = \|\phi_f(0) - \phi_f(-\frac{\pi}{x})\|_1.$$

Now let $|x|$ tend to infinity and the proposition follows. \square

REMARK. The fact that for every $f \in L^1$

$$(1.16) \quad \lim_{|x| \rightarrow \infty} \hat{f}(x) = 0$$

is usually known as the Riemann-Lebesgue lemma (RLL, for short).

From Proposition 1.3 one should not conclude that $\hat{f} \in L^1$ for every $f \in L^1$. Let, for example, $f(t) = e^{-t}$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. Then $\hat{f}(x) = (1 - ix)^{-1}$ and it is clear that $\hat{f} \notin L^1$.

We have shown that if $f \in L^1$ then \hat{f} is continuous on \mathbb{R} and \hat{f} vanishes at infinity. It seems natural to ask whether every function with these two properties is the Fourier transform of some function in L^1 . That this is not the case in general may be shown by the following example. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$(1.17) \quad g(x) = \begin{cases} (\log x)^{-1} & \text{if } x > e \\ \frac{x}{e} & \text{if } 0 \leq x \leq e \\ -g(-x) & \text{if } x < 0. \end{cases}$$

Clearly $g \in C_0(\mathbb{R})$. For a proof that g is not the Fourier transform of any $f \in L^1$ we refer the reader to GOLDBERG [1; pp. 8-9].

2. FOURIER-INVERSION (Recovering f from \hat{f})

We continue the general theory of Fourier transforms by investigating the question whether and how one can determine the function f if \hat{f} is given. We begin by recalling a theorem from the theory of Fourier series.

THEOREM. (Dirichlet) *If the function f is of bounded variation on the interval $[0, \delta]$ for some $\delta > 0$, then*

$$(2.1) \quad \lim_{R \rightarrow \infty} \frac{1}{\pi} \int_0^{\delta} f(t) \frac{\sin Rt}{t} dt = \frac{1}{2} \lim_{t \rightarrow 0} f(t) =: \frac{1}{2} f(+0).$$

PROOF. See any textbook on Fourier series. \square

As an easy consequence we have the following

THEOREM. (Jordan) *If $f \in L^1$ and if f is of bounded variation on some neighborhood of the point t_0 , then*

$$(2.2) \quad \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R e^{-ixt_0} \hat{f}(x) dx = \frac{f(t_0+0) + f(t_0-0)}{2}.$$

COROLLARY. *If $f \in L^1$ and f is of bounded variation on a neighborhood of t_0 and f is continuous at t_0 , then*

$$(2.3) \quad f(t_0) = \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-R}^R e^{-ixt_0} \hat{f}(x) dx.$$

We thus have found a set of conditions under which it is possible to recover f from the values of \hat{f} . In order to obtain more significant results concerning the Fourier inversion problem we introduce the following

DEFINITION. If the function g is integrable on $[-R, R]$ for every $R > 0$, then the integral $\int_{-\infty}^{\infty} g(x) dx$ is said to be Cesàro-integrable (or C-summable) to the value A if (compare p. 22)

$$(2.4) \quad \lim_{R \rightarrow \infty} \int_{-R}^R \left(1 - \frac{|x|}{R}\right) g(x) dx = A.$$

PROPOSITION 2.1. *The Cesàro-summability process is regular. In other words: If $g \in L^1$ and $\int_{-\infty}^{\infty} g(x) dx = A$, then $\int_{-\infty}^{\infty} g(x) dx$ is also C-summable to the value A .*

PROOF. For any $R > 0$ let g_R be defined by

$$(2.5) \quad g_R(x) = \begin{cases} (1 - \frac{|x|}{R}) g(x) & \text{if } |x| \leq R \\ 0 & \text{if } |x| > R. \end{cases}$$

Then

$$(2.6) \quad |g_R(x)| \leq |g(x)| \quad \text{for all } x \in \mathbf{R}$$

and

$$(2.7) \quad \lim_{R \rightarrow \infty} g_R(x) = g(x) \quad \text{for all } x \in \mathbf{R}.$$

Hence, by LDCT, we have

$$(2.8) \quad \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} g_R(x) dx = \int_{-\infty}^{\infty} g(x) dx = A$$

or, equivalently,

$$(2.9) \quad \lim_{R \rightarrow \infty} \int_{-R}^R (1 - \frac{|x|}{R}) g(x) dx = A$$

proving the proposition. \square

From the theory of Lebesgue-integration we borrow the following result: If f is integrable on $[-R, R]$ for every $R > 0$ then

$$(2.10) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h |f(x+t) - f(x)| dt = 0$$

for almost all $x \in \mathbf{R}$.

DEFINITION. The set of points for which (2.10) holds true is called the *Lebesgue set* of $f \in L^1$.

It is clear that the Lebesgue set of f contains all points at which f is continuous. It should be emphasized that if $f \in L^1$ then the complement of the Lebesgue set of f has measure 0.

THEOREM 2.1. If $f \in L^1$ and if u belongs to the Lebesgue set of f , then

$$(2.11) \quad \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R (1 - \frac{|x|}{R}) e^{-iux} \hat{f}(x) dx = f(u).$$

Hence, this inversion relation holds true for

- (a) almost all $u \in \mathbf{R}$
- (b) all points u at which f is continuous.

Before proving this theorem we state two important theorems which we will use on several occasions.

THEOREM. (Fubini) If $f \in L^1(\mathbb{R}^2)$, i.e. if

$$(2.12) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y)| \, dx dy$$

is finite, then $f(x_0, y)$ belongs to $L^1(\mathbb{R})$ for almost all $x_0 \in \mathbb{R}$. Moreover, the function $g : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$(2.13) \quad g(x) := \int_{-\infty}^{\infty} f(x,y) \, dy$$

belongs to $L^1(\mathbb{R})$ and

$$(2.14) \quad \int_{-\infty}^{\infty} g(x) \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx dy.$$

Similarly $f(x, y_0)$ belongs to $L^1(\mathbb{R})$ for almost all $y_0 \in \mathbb{R}$ and

$$(2.15) \quad \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x,y) \, dx \right\} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx dy.$$

THEOREM. (Tonelli-Hobson) If $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ is such that $f(x_0, y)$ belongs to $L^1(\mathbb{R})$ for almost all $x_0 \in \mathbb{R}$ (or if $f(x, y_0)$ belongs to $L^1(\mathbb{R})$ for almost all $y_0 \in \mathbb{R}$) and if

$$(2.16) \quad g(x) := \int_{-\infty}^{\infty} f(x,y) \, dy \quad (\text{or } h(y) := \int_{-\infty}^{\infty} f(x,y) \, dx)$$

belongs to $L^1(\mathbb{R})$, then $f \in L^1(\mathbb{R}^2)$ so that (by Fubini's theorem)

$$(2.17) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx dy = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x,y) \, dx \right\} dy = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x,y) \, dy \right\} dx.$$

PROOF OF THEOREM 2.1. For $R > 0$ and $u \in \mathbb{R}$ we define

$$(2.18) \quad \begin{aligned} S_R(u) &:= \frac{1}{2\pi} \int_{-R}^R \left(1 - \frac{|x|}{R}\right) e^{-iux} \hat{f}(x) \, dx = \\ &= \frac{1}{2\pi} \int_{-R}^R \left(1 - \frac{|x|}{R}\right) e^{-iux} \left(\int_{-\infty}^{\infty} e^{ixt} f(t) \, dt \right) dx = \\ &= \frac{1}{2\pi} \int_{-R}^R \left(\int_{-\infty}^{\infty} \left(1 - \frac{|x|}{R}\right) e^{-ix(u-t)} f(t) \, dt \right) dx. \end{aligned}$$

Now observe that for every $x_0 \in [-R, R]$

$$(2.19) \quad \left(1 - \frac{|x_0|}{R}\right) e^{-ix_0(u-t)} f(t)$$

belongs to L^1 as a function of t and that

$$(2.20) \quad \int_{-\infty}^{\infty} \left(1 - \frac{|x|}{R}\right) e^{-ix(u-t)} f(t) dt$$

(being a continuous function of x on $[-R, R]$) belongs to L^1 .
Hence, by the Tonelli-Hobson theorem, we obtain

$$(2.21) \quad S_R(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left(\int_{-R}^R \left(1 - \frac{|x|}{R}\right) e^{-ix(u-t)} dx \right) dt.$$

Since

$$(2.22) \quad \int_{-R}^R \left(1 - \frac{|x|}{R}\right) e^{-ix(u-t)} dx = 2 \frac{1 - \cos R(u-t)}{R(u-t)^2} \quad (\geq 0)$$

it follows that for all $u \in \mathbb{R}$

$$(2.23) \quad S_R(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1 - \cos R(u-t)}{R(u-t)^2} dt = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u-t) \frac{1 - \cos Rt}{Rt^2} dt.$$

This may also be written as

$$(2.24) \quad S_R(u) = \frac{1}{\pi} \left(\int_{-\infty}^0 + \int_0^{\infty} \right) f(u-t) \frac{1 - \cos Rt}{Rt^2} dt = \\ = \frac{1}{\pi} \int_0^{\infty} (f(u+t) + f(u-t)) \frac{1 - \cos Rt}{Rt^2} dt.$$

Since

$$(2.25) \quad \int_0^{\infty} \frac{1 - \cos Rt}{Rt^2} dt = \int_0^{\infty} \frac{1 - \cos v}{v^2} dv = \lim_{a \rightarrow \infty} \int_{\frac{1}{a}}^a \frac{1 - \cos v}{v^2} dv = \\ = \lim_{a \rightarrow \infty} \left(-\frac{1}{v} (1 - \cos v) \Big|_{\frac{1}{a}}^a + \int_{\frac{1}{a}}^a \frac{\sin v}{v} dv \right) = \int_0^{\infty} \frac{\sin v}{v} dv = \frac{\pi}{2}$$

it follows that for any positive δ

$$(2.26) \quad S_R(u) - f(u) = \frac{1}{\pi} \int_0^{\infty} (f(u+t) + f(u-t) - 2f(u)) \frac{1 - \cos Rt}{Rt^2} dt = \\ = \frac{1}{\pi} \left(\int_0^{\delta} + \int_{\delta}^{\infty} \right) (f(u+t) + f(u-t) - 2f(u)) \frac{1 - \cos Rt}{Rt^2} dt = \\ = I_1 + I_2, \text{ say.}$$

Defining

$$(2.27) \quad \phi_u(y) = |f(u+y) + f(u-y) - 2f(u)|$$

it is clear that ϕ_u is integrable over $[0, t]$ for every $t > 0$
so that we may define

$$(2.28) \quad \phi_u(t) = \int_0^t \phi_u(y) dy, \quad (t > 0).$$

Now suppose that u belongs to the Lebesgue set of f . Then

$$(2.29) \quad \lim_{t \rightarrow 0} \frac{1}{t} \phi_u(t) = \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t |f(u+y) + f(u-y) - 2f(u)| dy \leq \\ \leq \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t (|f(u+y)-f(u)| + |f(u-y)-f(u)|) dy = 0.$$

Hence, given $\varepsilon > 0$, we may choose δ such that

$$(2.30) \quad \frac{1}{t} \phi_u(t) \leq \varepsilon, \quad (0 < t \leq \delta)$$

or, equivalently,

$$(2.31) \quad \phi_u(t) \leq \varepsilon t, \quad (0 < t \leq \delta).$$

Choosing R such that $\frac{1}{R} < \delta$ we have

$$(2.32) \quad |\pi I_1| \leq \int_0^{\frac{\delta}{R}} |f(u+t) + f(u-t) - 2f(u)| \frac{1-\cos Rt}{Rt^2} dt = \\ = \left(\int_0^{\frac{1}{R}} + \int_{\frac{1}{R}}^{\frac{\delta}{R}} \right) \phi_u(t) \frac{1-\cos Rt}{Rt^2} dt = I_1' + I_1'', \text{ say.}$$

Since

$$(2.33) \quad I_1' \leq \int_0^{\frac{1}{R}} \phi_u(t) \frac{\frac{1}{2}(Rt)^2}{Rt^2} dt = \frac{R}{2} \int_0^{\frac{1}{R}} \phi_u(t) dt = \\ = \frac{R}{2} \phi_u\left(\frac{1}{R}\right) \leq \frac{\varepsilon}{2}$$

and

$$(2.34) \quad I_1'' \leq \int_{\frac{1}{R}}^{\frac{\delta}{R}} \phi_u(t) \frac{2}{Rt^2} dt = \phi_u(t) \frac{2}{Rt^2} \Big|_{\frac{1}{R}}^{\frac{\delta}{R}} + 4 \int_{\frac{1}{R}}^{\frac{\delta}{R}} \frac{\phi_u(t)}{Rt^3} dt = \\ = \frac{2\phi_u(\frac{\delta}{R})}{R\delta^2} - 2R\phi_u\left(\frac{1}{R}\right) + 4 \int_{\frac{1}{R}}^{\frac{\delta}{R}} \frac{\phi_u(t)}{Rt^3} dt \leq \\ \leq \frac{2\varepsilon}{R\delta} + 4\varepsilon \int_{\frac{1}{R}}^{\frac{\delta}{R}} \frac{dt}{Rt^2} < 2\varepsilon + 4\varepsilon = 6\varepsilon$$

it follows that

$$(2.35) \quad |\pi I_1| < \frac{\varepsilon}{2} + 6\varepsilon = \frac{13\varepsilon}{2}.$$

An estimate for I_2 is

$$(2.36) \quad |I_2| \leq \frac{2}{\pi R} \int_{\delta}^{\infty} \frac{\phi_u(t)}{t^2} dt$$

and since

$$(2.37) \quad \int_{\delta}^{\infty} \frac{\phi_u(t)}{t^2} dt < \infty$$

it clearly follows that $\lim_{R \rightarrow \infty} I_2 = 0$.

Combining our results we obtain

$$(2.38) \quad \limsup_{R \rightarrow \infty} |S_R(u) - f(u)| \leq \limsup_{R \rightarrow \infty} |I_1| + \limsup_{R \rightarrow \infty} |I_2| \leq \frac{13\epsilon}{2\pi}$$

and since $\epsilon > 0$ may be chosen as small as we please it follows that

$$(2.39) \quad \lim_{R \rightarrow \infty} S_R(u) = f(u)$$

for every u in the Lebesgue set of f , proving the theorem. \square

COROLLARY. If f and \hat{f} belong to L^1 and if f is continuous at u , then

$$(2.40) \quad f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{f}(x) dx.$$

PROOF. Since $\hat{f} \in L^1$ the above integral is certainly C-summable and

$$(2.41) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{f}(x) dx = \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-R}^R \left(1 - \frac{|x|}{R}\right) e^{-iux} \hat{f}(x) dx.$$

Since f is continuous at u , this point belongs to the Lebesgue set of f and the assertion follows by the previous theorem. \square

THEOREM 2.2. If $f \in L^1$ and $\hat{f} \equiv 0$ then $f(t) = 0$ for almost all $t \in \mathbb{R}$.

PROOF. By Theorem 2.1 it follows that $f(t) = 0$ for all t in the Lebesgue set of f . Hence, f can only differ from 0 on a set of measure 0. \square

COROLLARY. If $f, g \in L^1$ and if $\hat{f}(x) = \hat{g}(x)$ for all $x \in \mathbb{R}$, then $f(t) = g(t)$ for almost all $t \in \mathbb{R}$.

PROOF. The Fourier transform of $f-g$ is $\hat{f}-\hat{g} \equiv 0$. Hence $f(t) - g(t) = 0$ for almost all $t \in \mathbb{R}$. \square

COROLLARY. Two functions in L^1 which differ on a set of positive measure have distinct Fourier transforms.

THEOREM 2.3. If f is integrable over $[-R, R]$ for every $R > 0$ and if

$$(2.42) \quad \int_{-\infty}^{\infty} \frac{|f(t)|}{1+t^2} dt < \infty$$

then

$$(2.43) \quad \lim_{R \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1 - \cos R(u-t)}{R(u-t)^2} dt = f(u)$$

for almost all $u \in \mathbb{R}$.

PROOF. Case 1. $f \in L^1$

In this case we have (see the proof of Theorem 2.1)

$$(2.44) \quad S_R(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1 - \cos R(u-t)}{R(u-t)^2} dt \quad \text{for all } u \in \mathbb{R}.$$

Since

$$(2.45) \quad \lim_{R \rightarrow \infty} S_R(u) = f(u) \quad \text{for all } u \text{ in the Lebesgue set of } f,$$

and hence for almost all $u \in \mathbb{R}$, the proof is complete in case $f \in L^1$.

Case 2. For any $s > 0$ define f_s by

$$(2.46) \quad f_s(t) = \begin{cases} f(t) & \text{if } |t| \leq s \\ 0 & \text{if } |t| > s. \end{cases}$$

Then

$$(2.47) \quad |f_s(t)| \leq \frac{1+s^2}{1+t^2} |f(t)|, \quad \forall t \in \mathbb{R}$$

so that $f_s \in L^1$.

Hence, by case 1,

$$(2.48) \quad \lim_{R \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f_s(t) \frac{1 - \cos R(u-t)}{R(u-t)^2} dt = f_s(u)$$

for almost all $u \in \mathbb{R}$.

From this and the definition of f_s it is clear that

$$(2.49) \quad \lim_{R \rightarrow \infty} \int_{-s}^s f(t) \frac{1 - \cos R(u-t)}{R(u-t)^2} dt = \pi f(u)$$

for almost all $u \in [-s, s]$.

Now observe that, for $-s < u < s$,

$$(2.50) \quad \left| \int_{|t| \geq s} f(t) \frac{1 - \cos R(u-t)}{R(u-t)^2} dt \right| \leq \frac{2}{R} \int_{|t| \geq s} \frac{|f(t)|}{(u-t)^2} dt =$$

$$= \frac{2}{R} \int_{|t| \geq s} \frac{|f(t)|}{1+t^2} \frac{1+t^2}{(u-t)^2} dt \leq \frac{2K_s}{R} \int_{-\infty}^{\infty} \frac{|f(t)|}{1+t^2} dt$$

where K_s is a number satisfying

$$(2.51) \quad \frac{1+t^2}{(u-t)^2} \leq K_s, \quad (|t| \geq s).$$

It follows that for every $u \in (-s, s)$

$$(2.52) \quad \lim_{R \rightarrow \infty} \frac{1}{\pi} \int_{|t| \geq s} f(t) \frac{1 - \cos R(u-t)}{R(u-t)^2} dt = 0.$$

In combination with (2.48) we thus have

$$(2.53) \quad \lim_{R \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1 - \cos R(u-t)}{R(u-t)^2} dt = f(u)$$

for almost all $u \in (-s, s)$. Since s may be chosen as large as we please, the theorem follows. \square

3. CONVOLUTION PRODUCTS OF FUNCTIONS IN L^1

We recall that if f and g belong to $L^1(\mathbb{R})$ then the function

$$(3.1) \quad f(x-t)g(t), \quad (t \in \mathbb{R})$$

is measurable for every $x \in \mathbb{R}$. Moreover, this function is integrable (i.e. belongs to L^1) for almost all $x \in \mathbb{R}$, so that we can define

$$(3.2) \quad h(x) := \int_{-\infty}^{\infty} f(x-t)g(t) dt$$

for almost all $x \in \mathbb{R}$, namely for all those $x \in \mathbb{R}$ for which

$$(3.3) \quad \int_{-\infty}^{\infty} |f(x-t)g(t)| dt < \infty.$$

By means of Fubini's theorem it may be shown (see RUDIN [2; pp. 146-147]) that the function h belongs to L^1 . For h we will write $f * g$, the so called *convolution product* of f and g . The reader may verify that the binary operation $*$ satisfies the commutative and associative laws, i.e. for $f, g, h \in L^1$ we have

$$(3.4) \quad f * g = g * f$$

and

$$(3.5) \quad (f * g) * h = f * (g * h).$$

Another important property of the convolution product is

$$(3.6) \quad \|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1.$$

One of the most intriguing properties of the convolution product is expressed in the following

THEOREM 3.1. For $f, g \in L^1$ we have $(f * g)^\wedge = \hat{f} \cdot \hat{g}$.

*In more verbal language: The Fourier transform of the convolution product $f * g$ is equal to the pointwise product of the Fourier transforms of f and g .*

PROOF. This is a direct consequence of the Tonelli-Hobson theorem. For any $x \in \mathbb{R}$ we have

$$(3.7) \quad \begin{aligned} \hat{f}(x) \cdot \hat{g}(x) &= \left(\int_{-\infty}^{\infty} e^{ixt} f(t) dt \right) \left(\int_{-\infty}^{\infty} e^{ixu} g(u) du \right) = \\ &= \int_{-\infty}^{\infty} g(u) \left\{ \int_{-\infty}^{\infty} e^{ix(u+t)} f(t) dt \right\} du \end{aligned}$$

and since the Tonelli-Hobson theorem applies to the function

$$(3.8) \quad g(u)f(t)e^{ix(u+t)}, \quad ((u,t) \in \mathbb{R}^2)$$

the above iterated integral equals

$$(3.9) \quad \begin{aligned} \int_{-\infty}^{\infty} g(u) \left\{ \int_{-\infty}^{\infty} e^{ixt} f(t-u) dt \right\} du &= \\ &= \int_{-\infty}^{\infty} e^{ixt} \left\{ \int_{-\infty}^{\infty} f(t-u)g(u) du \right\} dt = \\ &= \int_{-\infty}^{\infty} e^{ixt} (f * g)(t) dt = (f * g)^\wedge(x) \end{aligned}$$

proving the theorem. \square

4. APPROXIMATE IDENTITIES

We note that there is no identity element (in L^1) with respect to the binary operation $*$. For if there were a $d \in L^1$ such that

$$(4.1) \quad d * f = f, \text{ for all } f \in L^1$$

then we would also have

$$(4.2) \quad d * d = d$$

and hence, by Theorem 3.1,

$$(4.3) \quad \hat{d} \cdot \hat{d} = \hat{d} \quad \text{or} \quad (\hat{d}(x))^2 = \hat{d}(x) \quad \text{for all } x \in \mathbb{R}.$$

It follows that \hat{d} can only assume values in the set $\{0,1\}$. However, we know that \hat{d} is continuous on \mathbb{R} and that $\lim_{|x| \rightarrow \infty} \hat{d}(x) = 0$ so that we must have $\hat{d}(x) = 0$ for all $x \in \mathbb{R}$. Hence, $d(x) = 0$ for almost all $x \in \mathbb{R}$; but then it is impossible to have $d * f = f$ for all $f \in L^1$.

Therefore we introduce the notion of *approximate identity* (app. id., for short).

DEFINITION. An approximate identity is a non-negative sequence $\{d_n\}_{n=1}^{\infty}$ in L^1 having the properties $\|d_n\|_1 = 1$, and $d_n * f \rightarrow f$, as $n \rightarrow \infty$, the limit being taken with respect to the usual L^1 -norm, i.e.

$$(4.4) \quad \lim_{n \rightarrow \infty} \|d_n * f - f\|_1 = 0.$$

In the sequel we will frequently make use of the following two functions (in L^1)

$$(4.5) \quad \Delta(t) = \begin{cases} 1-|t| & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

and

$$(4.6) \quad \delta(t) = \frac{1}{\pi} \frac{1-\cos t}{t^2}, \quad (t \in \mathbb{R}).$$

LEMMA 4.1. $\hat{\Delta} = 2\pi\delta$ and $\hat{\delta} = \Delta$.

PROOF. Clearly Δ is an even function so that

$$(4.7) \quad \begin{aligned} \hat{\Delta}(x) &= \int_{-\infty}^{\infty} e^{ixt} \Delta(t) dt = 2 \int_0^{\infty} \Delta(t) \cos xt dt = \\ &= 2 \int_0^1 (1-t) \cos xt dt = \frac{2(1-\cos x)}{x^2} = 2\pi\delta(x) \end{aligned}$$

proving the first assertion.

Next observe that $\Delta, \hat{\Delta} \in L^1$ and that Δ is continuous on \mathbb{R} so that by the corollary on page 41

$$(4.8) \quad \Delta(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{\Delta}(x) dx = \int_{-\infty}^{\infty} e^{-iux} \delta(x) dx.$$

Since $\Delta(u) = \Delta(-u)$ for all $u \in \mathbb{R}$ it follows that for all $u \in \mathbb{R}$

$$(4.9) \quad \Delta(u) = \int_{-\infty}^{\infty} e^{iux} \delta(x) dx$$

proving that $\Delta = \hat{\delta}$. \square

It will also be convenient to have at our disposal the functions

$$(4.10) \quad \Delta_R(x) := \Delta\left(\frac{x}{R}\right) \quad \text{and} \quad \delta_R(x) := R\delta(Rx) \quad \text{with } R > 0.$$

LEMMA 4.2. $\|\delta_R\|_1 = 1$ and $\widehat{\delta}_R = \Delta_R$.

PROOF. For every $R > 0$ we have

$$(4.11) \quad \|\delta_R\|_1 = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos Rt}{Rt^2} dt = 1$$

as was shown in the proof of Theorem 2.1.

Since $\Delta = \widehat{\delta}$ we have

$$(4.12) \quad \Delta\left(\frac{x}{R}\right) = \int_{-\infty}^{\infty} e^{i\frac{x}{R}t} \delta(t) dt = \int_{-\infty}^{\infty} e^{ixu} R\delta(Ru) du = \int_{-\infty}^{\infty} e^{ixu} \delta_R(u) du$$

proving the second assertion. \square

Note that Δ_R is a Fourier transform indeed.

Before proving one of our main results we recall the following

LEMMA. (Fatou) For any sequence $\{f_n\}_{n=1}^{\infty}$ of non-negative measurable functions on \mathbb{R} one has

$$(4.13) \quad \int_{-\infty}^{\infty} \liminf_{n \rightarrow \infty} f_n(x) dx \leq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx.$$

We are now able to prove that $\{\delta_n\}_{n=1}^{\infty}$ is an app. id..

THEOREM 4.1. For every $f \in L^1$ we have

$$(4.14) \quad \lim_{n \rightarrow \infty} \|\delta_n * f - f\|_1 = 0.$$

PROOF. For every positive integer n we have

$$(4.15) \quad (\delta_n * f)(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1 - \cos n(u-t)}{n(u-t)^2} dt$$

so that by Theorem 2.3

$$(4.16) \quad \lim_{n \rightarrow \infty} (\delta_n * f)(u) = f(u)$$

for almost all $u \in \mathbb{R}$.

Next observe that (applying Fatou's lemma to $f_n := \delta_n * f$)

$$(4.17) \quad \begin{aligned} \|f\|_1 &= \int_{-\infty}^{\infty} |f(u)| du = \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} |(\delta_n * f)(u)| du \leq \\ &\leq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} |(\delta_n * f)(u)| du = \liminf_{n \rightarrow \infty} \|\delta_n * f\|_1 \end{aligned}$$

from which it follows that

$$(4.18) \quad \|f\|_1 \leq \liminf_{n \rightarrow \infty} \|\delta_n * f\|_1.$$

Since $\|\delta_n\|_1 = 1$ and $\|\delta_n * f\|_1 \leq \|\delta_n\|_1 \cdot \|f\|_1$ it follows that

$$(4.19) \quad \limsup_{n \rightarrow \infty} \|\delta_n * f\|_1 \leq \|f\|_1.$$

Combining (4.18) and (4.19) we thus obtain

$$(4.20) \quad \lim_{n \rightarrow \infty} \|\delta_n * f\|_1 = \|f\|_1.$$

Since in addition

$$(4.21) \quad \lim_{n \rightarrow \infty} (\delta_n * f)(u) = f(u) \quad \text{for almost all } u \in \mathbb{R}$$

it follows that (for details see GOLDBERG [1; p. 4])

$$(4.22) \quad \lim_{n \rightarrow \infty} \|(\delta_n * f) - f\|_1 = 0$$

completing the proof. \square

5. EXISTENCE AND CONSTRUCTION OF CERTAIN FOURIER TRANSFORMS

In the previous section we saw that the function $\Delta_n \cdot \hat{f}$ (which vanishes outside the interval $(-n, n)$) is the Fourier transform of $\delta_n * f$ so that any $f \in L^1$ can be approximated arbitrarily close (in the L^1 -norm) by a function (in L^1) whose Fourier transform vanishes outside some bounded interval. Below we shall show that for any bounded interval $[a, b]$ there is a function in L^1 whose Fourier transform is identically 1 on $[a, b]$ and identically 0 outside a slightly larger interval.

THEOREM 5.1. *Given any real numbers a, b and h such that $a < b$, $h > 0$, there exists a $w \in L^1$ such that*

$$(5.1) \quad \begin{cases} \hat{w}(x) = 1 & \text{for } a \leq x \leq b \\ \hat{w}(x) = 0 & \text{for } x \leq a-h \text{ and } x \geq b+h \end{cases}$$

whereas w is linear on the intervals $[a-h, a]$ and $[b, b+h]$.

PROOF. Let $c = \frac{1}{2}(b-a)$. We already know that for every $R > 0$, Δ_R is a Fourier transform so that also

$$(5.2) \quad \frac{1}{h} \{ (c+h)\Delta_{c+h} - c\Delta_c \}$$

is the Fourier transform of some $w_1 \in L^1$. From the graphs of $(c+h)\Delta_{c+h}$ and $c\Delta_c$ it is easily seen that

$$(5.3) \quad \hat{w}_1(x) = \begin{cases} 1 & \text{for } -c \leq x \leq c \\ 0 & \text{for } x \leq -c-h \text{ and } x \geq c+h \end{cases}$$

and that \hat{w}_1 is linear on $[-c-h, -c]$ and $[c, c+h]$.

Now let

$$(5.4) \quad w(t) := e^{-\frac{1}{2}(a+b)it} w_1(t), \quad (t \in \mathbb{R})$$

so that by an easy computation

$$(5.5) \quad \hat{w}(x) = \hat{w}_1(x - \frac{1}{2}(a+b)).$$

Hence

$$(5.6) \quad \hat{w}(x) = 1 \quad \text{for} \quad -c \leq x - \frac{1}{2}(a+b) \leq c$$

which is equivalent to saying that (recall the definition of c)

$$(5.7) \quad \hat{w}(x) = 1 \quad \text{for} \quad a \leq x \leq b.$$

The reader will have no difficulty to complete the proof. \square

As an application we have

THEOREM 5.2. *If $f \in L^1$, $\hat{f}(0) = 0$ and $\varepsilon > 0$, then there exists an $h \in L^1$ satisfying the following three properties*

- (i) $\|h\|_1 < \varepsilon$
- (ii) $\hat{h}(x) = \hat{f}(x)$ for all x in some nbhd of 0
- (iii) $\hat{h}(x) = 0$ if $\hat{f}(x) = 0$.

PROOF. By the previous theorem there exists a $w \in L^1$ such that

$$(5.8) \quad \hat{w}(x) = 1 \quad \text{for} \quad |x| \leq 1.$$

Defining

$$(5.9) \quad w_R(t) := R w(Rt), \quad (R > 0; t \in \mathbb{R})$$

an easy computation shows that

$$(5.10) \quad \hat{w}_R(x) = \hat{w}\left(\frac{x}{R}\right)$$

so that

$$(5.11) \quad \hat{w}_R(x) = 1 \quad \text{for} \quad |x| \leq R.$$

Now observe that

$$(5.12) \quad \begin{aligned} (w_R * f)(x) &= \int_{-\infty}^{\infty} w_R(x-t)f(t) dt = \\ &= \int_{-\infty}^{\infty} w_R(x-t)f(t) dt - w_R(x) \int_{-\infty}^{\infty} f(t) dt = \end{aligned}$$

... (since $\hat{f}(0) = 0$) ...

$$= \int_{-\infty}^{\infty} f(t) \{w_R(x-t) - w_R(x)\} dt$$

so that

$$\begin{aligned}
 (5.13) \quad \|w_R * f\|_1 &= \int_{-\infty}^{\infty} |(w_R * f)(x)| dx \leq \\
 &\leq \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(t)| |w_R(x-t) - w_R(x)| dt \right\} dx = \\
 &\quad \dots \text{(by the Tonelli-Hobson theorem)} \dots \\
 &= \int_{-\infty}^{\infty} |f(t)| \left\{ \int_{-\infty}^{\infty} |w_R(x-t) - w_R(x)| dx \right\} dt = \\
 &= \int_{-\infty}^{\infty} |f(t)| \left\{ \int_{-\infty}^{\infty} R |w(Rx-Rt) - w(Rx)| dx \right\} dt = \\
 &= \int_{-\infty}^{\infty} |f(t)| \left\{ \int_{-\infty}^{\infty} |w(u-Rt) - w(u)| du \right\} dt.
 \end{aligned}$$

In combination with the observation that

$$(5.14) \quad |f(t)| \int_{-\infty}^{\infty} |w(u-Rt) - w(u)| du \leq 2\|w\|_1 \cdot |f(t)|$$

and

$$(5.15) \quad \lim_{R \rightarrow 0} \int_{-\infty}^{\infty} |w(u-Rt) - w(u)| du = 0$$

it follows from LDCT that

$$(5.16) \quad \lim_{R \rightarrow 0} \|w_R * f\|_1 = 0.$$

Hence, we may choose R so small that

$$(5.17) \quad \|w_R * f\|_1 < \varepsilon$$

and for this R we define

$$(5.18) \quad h := w_R * f.$$

Then we certainly have $\|h\|_1 < \varepsilon$ and since

$$(5.19) \quad \hat{h} = \hat{w}_R \cdot \hat{f} \quad \text{and} \quad \hat{w}_R(x) = 1 \quad \text{for} \quad |x| \leq R$$

it follows that

$$(5.20) \quad \hat{h}(x) = \hat{f}(x) \quad \text{for} \quad |x| \leq R.$$

From (5.19) it is also clear that $\hat{h}(x) = 0$ if $\hat{f}(x) = 0$, completing the proof. \square

For later use we prove

THEOREM 5.3. *There exists a $g \in L^1$ such that*

$$(5.21) \quad \begin{cases} \hat{g}(x) > 0 & \text{if } x > 0 \\ \hat{g}(x) = 0 & \text{if } x \leq 0. \end{cases}$$

PROOF. Define the functions g and G by

$$(5.22) \quad g(t) := \frac{1}{2\pi} \frac{1}{(1+it)^2}, \quad (t \in \mathbb{R})$$

$$(5.23) \quad G(x) := \begin{cases} xe^{-x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then g and G belong to L^1 and integration by parts yields

$$(5.24) \quad \hat{G}(x) = \int_0^{\infty} te^{-t(1-ix)} dt = \frac{1}{(1-ix)^2} = 2\pi g(-x).$$

Observing that G and \hat{G} belong to L^1 and that G is continuous on \mathbb{R} , it follows by the corollary on page 41 that for all $t \in \mathbb{R}$

$$(5.25) \quad \begin{aligned} G(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \hat{G}(x) dx = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} 2\pi g(-x) dx = \int_{-\infty}^{\infty} e^{ixt} g(x) dx = \hat{g}(t) \end{aligned}$$

and the theorem follows. \square

As a consequence we have

THEOREM 5.4. *For any interval of the form $(-\infty, a]$, or $[a, \infty)$, there exists an $h \in L^1$ such that \hat{h} vanishes on the interval and does not vanish outside the interval in question.*

PROOF. Let g be as in the previous theorem so that

$$(5.26) \quad \hat{g}(x-a) = \int_{-\infty}^{\infty} e^{ixt} \{e^{-iat} g(t)\} dt.$$

Define

$$(5.27) \quad h_1(t) := e^{-iat} g(t), \quad (t \in \mathbb{R})$$

so that $h_1 \in L^1$ and $\hat{h}_1(x) = \hat{g}(x-a)$ for all $x \in \mathbb{R}$.

It follows from the previous theorem that $\hat{h}_1(x) = 0$ if $x \leq a$ and that $\hat{h}_1(x) > 0$ if $x > a$.

Defining

$$(5.28) \quad h_2(t) := e^{-iat} g(-t), \quad (t \in \mathbb{R})$$

we have

$$(5.29) \quad \hat{h}_2(x) = \hat{g}(a-x) \quad \text{for all } x \in \mathbb{R}$$

and the theorem follows. \square

REFERENCES

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2. RUDIN, W., *Real and Complex Analysis*, McGraw-Hill, 1966.

CHAPTER 7

ANALYTIC FUNCTIONS OF FOURIER TRANSFORMS

It is clear that if $f \in L^1$ and $n \in \mathbb{N}$ (\mathbb{N} denoting the set of all positive integers), then

$$(1) \quad (\hat{f})^n = (f * f * \dots * f)^{\wedge}$$

the convolution product consisting of n factors.

More generally we have that if

$$(2) \quad P(z) = \sum_{r=1}^n a_r z^r, \quad (a_r \in \mathbb{C})$$

is a polynomial such that $P(0) = 0$, then P takes Fourier transforms into Fourier transforms, i.e. if $f \in L^1$ then $P(\hat{f})$ is the Fourier transform of some $g \in L^1$, namely

$$(3) \quad g = \sum_{r=1}^n a_r \cdot (f)^{*r}$$

where

$$(4) \quad (f)^{*r} := f * f * \dots * f$$

the convolution product consisting of r factors.

In order to obtain a more general result we first prove

THEOREM 1. *Let $\phi(z)$ be analytic for $|z| < \epsilon$ for some $\epsilon > 0$ and let $\phi(0) = 0$. If $h \in L^1$ is such that $\|h\|_1 < \epsilon$, then $\phi(\hat{h})$ is a Fourier transform, i.e. there exists a $g \in L^1$ such that*

$$(5) \quad \phi(\hat{h}(x)) = \hat{g}(x) \quad \text{for all } x \in \mathbb{R}.$$

PROOF. Since ϕ is analytic and $\phi(0) = 0$ we may write

$$(6) \quad \phi(z) = \sum_{k=1}^{\infty} a_k z^k \quad \text{for } |z| < \epsilon$$

this series being absolutely convergent.

We also have

$$(7) \quad |\hat{h}(x)| \leq \|h\|_1 \quad \text{for all } x \in \mathbb{R}$$

so that

$$(8) \quad \phi(\hat{h}(x)) = \sum_{k=1}^{\infty} a_k (\hat{h}(x))^k.$$

Define $h_1 := h$ and, for $k = 2, 3, 4, \dots$, let $h_k := h_{k-1} * h$. Then

$$(9) \quad \|h_k\|_1 \leq \|h\|_1^k$$

and

$$(10) \quad \widehat{h}_k = (h^{*k})^\wedge = (\widehat{h})^k$$

so that

$$(11) \quad \widehat{h}_k(x) = (\widehat{h}(x))^k \quad \text{for all } x \in \mathbb{R}.$$

Now observe that

$$(12) \quad \left\| \sum_{k=m}^n a_k h_k \right\|_1 \leq \sum_{k=m}^n |a_k| \|h_k\|_1 \leq \sum_{k=m}^n |a_k| \|h\|_1^k.$$

Since $\|h\|_1 < \varepsilon$, the series

$$(13) \quad \sum_{k=1}^{\infty} |a_k| \|h\|_1^k$$

converges, so that

$$(14) \quad \sum_{k=m}^n |a_k| \|h\|_1^k \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

By the triangle inequality for the L^1 -norm it follows that also

$$(15) \quad \left\| \sum_{k=m}^n a_k h_k \right\|_1 \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Consequently the sequence

$$(16) \quad \left\{ \sum_{k=1}^n a_k h_k \right\}_{n=1}^{\infty}$$

is a Cauchy sequence in (the complete normed space) L^1 and hence is convergent with limit $g \in L^1$, say.

Now recall that

$$(17) \quad \|\widehat{f}_n - \widehat{f}\|_{\infty} = \|(f_n - f)^\wedge\|_{\infty} \leq \|f_n - f\|_1$$

so that in our situation

$$(18) \quad \left(\sum_{k=1}^n a_k h_k \right)^\wedge(x) \rightarrow \widehat{g}(x)$$

uniformly in x , as $n \rightarrow \infty$.

It follows that

$$(19) \quad \widehat{g}(x) = \sum_{k=1}^{\infty} a_k \widehat{h}_k(x) = \sum_{k=1}^{\infty} a_k \cdot (\widehat{h})^k(x) =$$

$$= \sum_{k=1}^{\infty} a_k \cdot (\hat{h}(x))^k = \phi(\hat{h}(x))$$

so that

$$(20) \quad \hat{g} = \phi(\hat{h})$$

and the proof is complete. \square

COROLLARY. If ϕ is analytic in the entire complex plane and $\phi(0) = 0$, then ϕ takes Fourier transforms into Fourier transforms. In other words, if $f \in L^1$, then there exists a $g \in L^1$ such that

$$(21) \quad \phi(\hat{f}(x)) = \hat{g}(x) \quad \text{for all } x \in \mathbf{R}.$$

The hypothesis $\phi(0) = 0$ in the previous theorem (and corollary) is essential. For if

$$(22) \quad Q(z) = a_0 + P(z) = a_0 + a_1 z + \dots + a_n z^n, \quad (a_0 \neq 0)$$

then for no $f \in L^1$ it is true that $Q(\hat{f})$ is a Fourier transform. For if it were we would have (since $P(\hat{f})$ is a Fourier transform indeed)

$$(23) \quad \lim_{|x| \rightarrow \infty} Q(\hat{f}(x)) = a_0 + \lim_{|x| \rightarrow \infty} P(\hat{f}(x)) = a_0 \neq 0$$

so that (by RLL) $Q(\hat{f})$ cannot be a Fourier transform.

Previously we showed that, given an interval $[a, b]$, there exists a $w \in L^1$ such that $\hat{w}(x) = 1$ for all $x \in [a, b]$. Hence, if Q is as above, then for every $f \in L^1$

$$(24) \quad Q(\hat{f}(x)) = a_0 \hat{w}(x) + P(\hat{f}(x)) \quad \text{for all } x \in [a, b]$$

so that $Q(\hat{f})$ coincides on $[a, b]$ with some Fourier transform, namely $a_0 \hat{w} + P(\hat{f})$.

This result will be generalized in the following

THEOREM 2. Let $\phi(z)$ be analytic at each point of some open connected set D of the complex plane and let $[a, b]$ be a closed bounded interval in \mathbf{R} . Then, if $f \in L^1$ and $\hat{f}(x) \in D$ for all $x \in [a, b]$, there exists a $g \in L^1$ such that

$$(25) \quad \phi(\hat{f}(x)) = \hat{g}(x) \quad \text{for all } x \in [a, b].$$

In order to prove this theorem we first establish four lemmas.

LEMMA 1. If $f \in L^1$, $\hat{f}(0) = 0$ and ϕ is analytic at $z=0$ such that $\phi(0) = 0$, then there exists a $g \in L^1$ such that

$$(26) \quad \phi(\hat{f}(x)) = \hat{g}(x)$$

for all x in some neighborhood of $x = 0$.

PROOF. By hypothesis there exists an $\varepsilon > 0$ such that $\phi(z)$ is analytic for $|z| < \varepsilon$. By Theorem 5.2 in Chapter 6 there exists an $h \in L^1$ such that

$$(27) \quad \|h\|_1 < \varepsilon \quad \text{and} \quad \hat{f}(x) = \hat{h}(x)$$

for all x in some neighborhood (N_0 , say) of 0.

Since $\|h\|_1 < \varepsilon$ Theorem 1 guarantees the existence of a $g \in L^1$ such that

$$(28) \quad \phi(\hat{h}(x)) = \hat{g}(x) \quad \text{for all } x \in \mathbb{R}.$$

Thus, for $x \in N_0$ we have

$$(29) \quad \phi(\hat{f}(x)) = \phi(\hat{h}(x)) = \hat{g}(x)$$

proving the lemma. \square

LEMMA 2. If $f \in L^1$, $\hat{f}(0) = 0$ and $\phi(z)$ is analytic at $z=0$, then there exists a $g \in L^1$ such that

$$(30) \quad \phi(\hat{f}(x)) = \hat{g}(x)$$

for all x in some neighborhood of 0.

PROOF. The case $\phi(0) = 0$ has been dealt with in the previous lemma. Therefore we assume $\phi(0) \neq 0$ and define

$$(31) \quad \psi(z) := \phi(z) - \phi(0).$$

Then $\psi(0) = 0$ and by Lemma 1 there exists a $g_1 \in L^1$ such that

$$(32) \quad \psi(\hat{f}(x)) = \hat{g}_1(x)$$

for all x in some neighborhood N_0 of 0. Without loss of generality we may assume that N_0 is a closed bounded interval so that by Theorem 5.1 in Chapter 6 there exists a $w \in L^1$ such that $\hat{w}(x) = 1$ for all $x \in N_0$. Now define g by

$$(33) \quad g := g_1 + \phi(0) \cdot w$$

and a simple calculation shows that for all $x \in N_0$ we have

$$(34) \quad \begin{aligned} \phi(\widehat{f}(x)) &= \psi(\widehat{f}(x)) + \phi(0) = \\ &= \psi(\widehat{f}(x)) + \phi(0) \cdot \widehat{w}(x) = \widehat{g}_1(x) + \phi(0) \cdot \widehat{w}(x) = \widehat{g}(x) \end{aligned}$$

proving the lemma. \square

LEMMA 3. If $f \in L^1$, $\widehat{f}(0) = \beta$ and $\phi(z)$ is analytic at $z=\beta$, then there exists a $g \in L^1$ such that

$$(35) \quad \phi(\widehat{f}(x)) = \widehat{g}(x)$$

for all x in some neighborhood of 0 .

PROOF. The case $\beta = 0$ has been dealt with in the previous lemma. In case $\beta \neq 0$ we choose $w \in L^1$ such that $\widehat{w}(x) = 1$ for all $x \in [-1, 1]$ and define $f_1 := f - \beta \cdot w$ and $\psi(z) := \phi(z+\beta)$. Then

$$(36) \quad \widehat{f}_1(0) = \widehat{f}(0) - \beta \widehat{w}(0) = \beta - \beta \cdot 1 = 0$$

and since ψ is analytic at $z=0$, Lemma 2 guarantees the existence of a $g \in L^1$ such that

$$(37) \quad \psi(\widehat{f}_1(x)) = \widehat{g}(x)$$

for all x in some neighborhood N_0 of 0 . Since we may assume that $N_0 \subset [-1, 1]$ it follows that for all $x \in N_0$

$$(38) \quad \begin{aligned} \phi(\widehat{f}(x)) &= \psi(\widehat{f}(x) - \beta) = \psi(\widehat{f}(x) - \beta \widehat{w}(x)) = \\ &= \psi(\widehat{f}_1(x)) = \widehat{g}(x) \end{aligned}$$

proving the lemma. \square

LEMMA 4. If $f \in L^1$, $\widehat{f}(\alpha) = \beta$ and $\phi(z)$ is analytic at $z=\beta$, then there exists a $g \in L^1$ such that

$$(39) \quad \phi(\widehat{f}(x)) = \widehat{g}(x)$$

for all x in some neighborhood of α .

PROOF. If $\alpha = 0$ we are done by the previous lemma.

If $\alpha \neq 0$ we define $f_1 \in L^1$ by $f_1(t) := e^{i\alpha t} f(t)$ so that

$$(40) \quad \widehat{f}_1(x) = \widehat{f}(x + \alpha) \quad \text{for all } x \in \mathbb{R}$$

and, in particular,

$$(41) \quad \widehat{f}_1(0) = \widehat{f}(\alpha) = \beta.$$

Hence, by Lemma 3, there exists a $g_1 \in L^1$ such that

$$(42) \quad \phi(\widehat{f}_1(x)) = \widehat{g}_1(x)$$

for all x in some neighborhood N_0 of 0. Hence

$$(43) \quad \phi(\hat{f}_1(x-\alpha)) = \hat{g}_1(x-\alpha)$$

for all x in a neighborhood N_α of α . Defining $g \in L^1$ by

$$(44) \quad g(t) := e^{-i\alpha t} g_1(t)$$

we obtain

$$(45) \quad \hat{g}(x) = \hat{g}_1(x-\alpha)$$

and it follows that

$$(46) \quad \phi(\hat{f}(x)) = \phi(\hat{f}_1(x-\alpha)) = \hat{g}_1(x-\alpha) = \hat{g}(x)$$

for all x in a neighborhood of α , proving the lemma. \square

We are now ready for the

PROOF OF THEOREM 2. By Lemma 4 each $x \in [a, b]$ is contained in an open interval on which $\phi(\hat{f})$ coincides with some Fourier transform. Since $[a, b]$ is compact, a finite number of these intervals will cover $[a, b]$ and we may assume that none of these intervals is entirely contained in one of the others. Now let (α_1, β_1) and (α_2, β_2) be any two of these intervals which have a point in common and suppose that $\alpha_1 < \alpha_2 < \beta_1 < \beta_2$. Choose g_1 and g_2 in L^1 such that

$$(47) \quad \begin{cases} \phi(\hat{f}(x)) = \hat{g}_1(x), & (\alpha_1 < x < \beta_1) \\ \phi(\hat{f}(x)) = \hat{g}_2(x), & (\alpha_2 < x < \beta_2) \end{cases}$$

and note that $\hat{g}_1(x) = \hat{g}_2(x)$ for all $x \in [\alpha_2, \beta_1]$.

Now choose w_1 and w_2 in L^1 such that

$$(48) \quad \begin{aligned} \hat{w}_1(x) &= 1, & (\alpha_1 \leq x \leq \alpha_2) \\ \hat{w}_1(x) &= 0, & (\beta_1 \leq x \leq \beta_2) \\ \hat{w}_2(x) &= 1, & (\beta_1 \leq x \leq \beta_2) \\ \hat{w}_2(x) &= 0, & (\alpha_1 \leq x \leq \alpha_2) \end{aligned}$$

and such that \hat{w}_1 and \hat{w}_2 are linear on $[\alpha_2, \beta_1]$.

Now define $h \in L^1$ by

$$(49) \quad h := w_1 * g_1 + w_2 * g_2.$$

Then we have

- (i) if $x \in (\alpha_1, \alpha_2)$, then

$$\hat{h}(x) = \hat{w}_1(x)\hat{g}_1(x) + \hat{w}_2(x)\hat{g}_2(x) =$$

$$= \hat{w}_1(x)\hat{g}_1(x) = \hat{g}_1(x) = \phi(\hat{f}(x)).$$
- (ii) if $x \in [\alpha_2, \beta_1]$, then

$$\hat{g}_1(x) = \hat{g}_2(x) \quad \text{and} \quad \hat{w}_1(x) + \hat{w}_2(x) = 1$$
so that for these values of x

$$\hat{h}(x) = \hat{g}_1(x) = \phi(\hat{f}(x)).$$
- (iii) if $x \in (\beta_1, \beta_2)$, then

$$\hat{h}(x) = \hat{w}_2(x)\hat{g}_2(x) = \hat{g}_2(x) = \phi(\hat{f}(x)).$$

It follows that for all $x \in (\alpha_1, \beta_2)$

$$(50) \quad \hat{h}(x) = \phi(\hat{f}(x))$$

so that $\phi(\hat{f}(x))$ coincides with a Fourier transform on (α_1, β_2) . Repeating this argument a finite number of times we are done. \square

Note that the only property of $[a, b]$ used in the proof was the fact that from any covering of $[a, b]$ by open sets we may extract a finite number of intervals which still form a covering of this interval. Thus, Theorem 2 remains true if we replace $[a, b]$ by any set of real numbers having this property, i.e. any compact set in \mathbb{R} . Consequently we have the following

COROLLARY. (Wiener) If $f \in L^1$ and $\hat{f}(x) \neq 0$ for all values of x belonging to some compact set K , then there exists a $g \in L^1$ such that

$$(51) \quad \frac{1}{\hat{f}(x)} = \hat{g}(x) \quad \text{for all } x \in K.$$

This corollary will play a crucial role in the proof of Wiener's general Tauberian theorem enunciated in Chapter 5.

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CHAPTER 8

WIENER'S GENERAL TAUBERIAN THEOREM

1. SOME SIMPLE PROPERTIES OF T_f AND \overline{T}_f

For any $f \in L^1$ let T_f be the set of all finite linear combinations of translates of f , i.e. $h \in T_f$ if and only if

$$(1.1) \quad h(x) = \sum_k a_k f(x+c_k) \quad \text{for all } x \in \mathbb{R}$$

for some finite set of real numbers c_k and complex a_k .

As usual let \overline{T}_f be the closure (under the L^1 -norm) of T_f in L^1 .

PROPOSITION 1.1. If g_1 and g_2 belong to \overline{T}_f then also $a_1 g_1 + a_2 g_2$ belongs to \overline{T}_f for any complex numbers a_1 and a_2 . In other words: \overline{T}_f is a linear space over the complex field \mathbb{C} .

PROPOSITION 1.2. If $g \in \overline{T}_f$ then, for any real c , also the translate g_c belongs to \overline{T}_f . In other words: \overline{T}_f is translation invariant.

PROPOSITION 1.3. If $g \in \overline{T}_f$ then $\overline{T}_g \subset \overline{T}_f$.

The proofs of these propositions are simple and are left to the reader.

2. PROOF OF WIENER'S THEOREM

THEOREM 2.1. If $f \in L^1$ and $\hat{f}(\lambda) = 0$, then $\hat{g}(\lambda) = 0$ for every $g \in \overline{T}_f$.

PROOF. First let $g \in T_f$, so that for all $x \in \mathbb{R}$

$$(2.1) \quad g(t) = \sum_k a_k f(t+c_k).$$

Then

$$(2.2) \quad \begin{aligned} \hat{g}(x) &= \sum_k a_k \int_{-\infty}^{\infty} e^{itx} f(t+c_k) dt = \\ &= \sum_k a_k \int_{-\infty}^{\infty} e^{i(u-c_k)x} f(u) du = \sum_k a_k e^{-ic_k x} \hat{f}(u) \end{aligned}$$

so that $\hat{g}(\lambda) = 0$.

Now let $g \in \overline{T}_f$. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in T_f such that

$$(2.3) \quad \lim_{n \rightarrow \infty} \|g_n - g\|_1 = 0$$

and for the corresponding Fourier transforms we have

$$(2.4) \quad \lim_{n \rightarrow \infty} \|\hat{g}_n - \hat{g}\|_\infty = 0$$

so that in particular

$$(2.5) \quad \lim_{n \rightarrow \infty} \hat{g}_n(\lambda) = \hat{g}(\lambda).$$

Since $g_n \in T_f$ we have $\hat{g}_n(\lambda) = 0$ and it follows that $\hat{g}(\lambda) = 0$, completing the proof. \square

In combination with Theorem 5.1 in Chapter 6 it follows that the statement $\overline{T}_f = L^1$ can only be true if $\hat{f}(x) \neq 0$ for all $x \in \mathbb{R}$. In other words: The condition " $\hat{f}(x) \neq 0$ for all $x \in \mathbb{R}$ " is necessary for Wiener's general Tauberian theorem. In order to prove that this condition is also sufficient we will make use of the following striking

THEOREM 2.2. *If $f \in L^1$ and $g \in \overline{T}_f$, then*

$$(2.6) \quad g * h \in \overline{T}_f$$

for every $h \in L^1$. In other words: \overline{T}_f is a (closed) ideal in the ring $(L^1, +, *)$.

PROOF. We may assume that neither g nor h is a zero function. Define $H := g * h$, so that for almost all $x \in \mathbb{R}$

$$(2.7) \quad H(x) = \int_{-\infty}^{\infty} g(x-t)h(t) dt.$$

Given $\varepsilon > 0$, choose N such that

$$(2.8) \quad \int_{|t| \geq N} |h(t)| dt < \frac{\varepsilon}{2\|g\|_1}$$

and let

$$(2.9) \quad H_N(x) := \int_{-N}^N g(x-t)h(t) dt$$

for all those x for which $H(x)$ is finite.

Then

$$(2.10) \quad H(x) - H_N(x) = \int_{|t| \geq N} g(x-t)h(t) dt$$

so that

$$\begin{aligned}
(2.11) \quad \|H - H_N\|_1 &\leq \int_{-\infty}^{\infty} \left\{ \int_{|t| \geq N} |g(x-t)h(t)| dt \right\} dx = \\
&= \int_{|t| \geq N} |h(t)| \left\{ \int_{-\infty}^{\infty} |g(x-t)| dx \right\} dt.
\end{aligned}$$

It follows that

$$(2.12) \quad \|H - H_N\|_1 < \|g\|_1 \frac{\epsilon}{2\|g\|_1} = \frac{\epsilon}{2}.$$

Now observe that there exists a $\delta > 0$ such that

$$(2.13) \quad \int_{-\infty}^{\infty} |g(x-y) - g(x)| dx \leq \frac{\epsilon}{2\|h\|_1}, \quad (|y| \leq \delta)$$

and choose t_0, t_1, \dots, t_n such that

$$(2.14) \quad -N = t_0 < t_1 < t_2 < \dots < t_n = N$$

and

$$(2.15) \quad t_k - t_{k-1} \leq \delta \quad \text{for } k = 1, 2, \dots, n.$$

Then we have

$$(2.16) \quad H_N(x) = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} g(x-t)h(t) dt.$$

Defining h_N by

$$(2.17) \quad h_N(x) := \sum_{k=1}^n g(x-t_k) \int_{t_{k-1}}^{t_k} h(t) dt$$

it is clear that $h_N \in T_g$ and

$$(2.18) \quad H_N(x) - h_N(x) = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \{g(x-t) - g(x-t_k)\} h(t) dt$$

so that

$$\begin{aligned}
(2.19) \quad \|H_N - h_N\|_1 &\leq \sum_{k=1}^n \int_{-\infty}^{\infty} dx \int_{t_{k-1}}^{t_k} |g(x-t) - g(x-t_k)| \cdot |h(t)| dt = \\
&= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} |h(t)| dt \int_{-\infty}^{\infty} |g(x-t) - g(x-t_k)| dx \leq \\
&\leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} |h(t)| dt \frac{\epsilon}{2\|h\|_1} = \frac{\epsilon}{2\|h\|_1} \int_{-N}^N |h(t)| dt \leq \frac{\epsilon}{2}.
\end{aligned}$$

It follows that

$$(2.20) \quad \|H - h_N\|_1 \leq \|H - H_N\|_1 + \|H_N - h_N\|_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and since $\varepsilon > 0$ may be chosen as small as we please, and $h_N \in T_g$ we find that $H \in \overline{T}_g$. By hypothesis $g \in \overline{T}_f$ so that by Proposition 1.3, $\overline{T}_g \subset \overline{T}_f$ and it follows that $H \in \overline{T}_f$, completing the proof. \square

Another important ingredient for Wiener's theorem is

THEOREM 2.3. If $f \in L^1$ and if for all $n \in \mathbb{N}$

$$(2.21) \quad \delta_n \in \overline{T}_f$$

where δ_n is as on page 46, then $\overline{T}_f = L^1$.

PROOF. Since $\delta_n \in \overline{T}_f$ we have by the previous theorem that for every $h \in L^1$

$$(2.22) \quad \delta_n * h \in \overline{T}_f.$$

Previously we found that $\lim_{n \rightarrow \infty} \|\delta_n * h - h\|_1 = 0$ for every $h \in L^1$. Combining these two facts we obtain

$$(2.23) \quad h \in \overline{T}_f \quad \text{for every } h \in L^1$$

and since

$$(2.24) \quad \overline{\overline{T}_f} = \overline{T}_f$$

it follows that

$$(2.25) \quad L^1 \subset \overline{T}_f$$

from which it is clear that $L^1 = \overline{T}_f$, completing the proof. \square

The finishing touch will be achieved by

THEOREM 2.4. If $f \in L^1$ and $\hat{f}(x) \neq 0$ for all $x \in \mathbb{R}$ then $\delta_n \in \overline{T}_f$ for all $n \in \mathbb{N}$.

PROOF. Fix any $n \in \mathbb{N}$. Then $\hat{f}(x) \neq 0$ for $-n \leq x \leq n$. Hence, there exists a $g_n \in L^1$ such that

$$(2.26) \quad \frac{1}{\hat{f}(x)} = \hat{g}_n(x), \quad (-n \leq x \leq n).$$

With $\Delta_n = \hat{\delta}_n$ as before we have for all $x \in \mathbb{R}$

$$(2.27) \quad \frac{\Delta_n(x)}{\hat{f}(x)} = \Delta_n(x) \hat{g}_n(x)$$

so that for all $x \in \mathbb{R}$

$$\hat{\delta}_n(x) = \hat{f}(x) \cdot \hat{\delta}_n(x) \cdot \hat{g}_n(x)$$

and hence

$$(2.28) \quad \widehat{\delta}_n = \widehat{f} \cdot \widehat{\delta}_n \cdot \widehat{g}_n .$$

From this it follows that (see the first corollary of Theorem 2.2 in Chapter 6)

$$(2.29) \quad \delta_n = f * (\delta_n * g_n)$$

and, since

$$(2.30) \quad f \in \overline{T}_f \quad \text{and} \quad \delta_n * g_n \in L^1$$

it follows from Theorem 2.2 that $\delta_n \in \overline{T}_f$, proving the theorem. \square

Finally we have

THEOREM 2.4. (Wiener's general Tauberian theorem)

If $f \in L^1$, then

$$(2.31) \quad \overline{T}_f = L^1$$

if and only if

$$(2.32) \quad \widehat{f}(x) \neq 0 \quad \text{for all } x \in \mathbb{R} .$$

PROOF. It has been shown before that condition (2.32) is necessary. That it is also sufficient is an easy consequence of the foregoing theorems. Indeed, since $\widehat{f}(x) \neq 0$ for all $x \in \mathbb{R}$, we have

$$(2.33) \quad \delta_n \in \overline{T}_f \quad \text{for all } n \in \mathbb{N}$$

so that by Theorem 2.3

$$(2.34) \quad \overline{T}_f = L^1 . \quad \square$$

LITERATURE

(See page 58)

CHAPTER 9

SOME ALGEBRAIC REFORMULATIONS

In this chapter we will reformulate some of the preceding results in terms of ideals in a commutative Banach algebra.

First we recall that a *complex algebra* is a vector space A over the complex field \mathbb{C} , on which an associative and distributive binary operation $*$ is defined, i.e.

$$(i) \quad f * (g * h) = (f * g) * h$$

$$(ii) \quad (f + g) * h = f * h + g * h$$

$$(iii) \quad f * (g + h) = f * g + f * h$$

for all f, g and h in A , whereas $*$ is related to scalar multiplication in such a way that for all $\alpha \in \mathbb{C}$ and all $f, g \in A$

$$(iv) \quad \alpha(f * g) = f * (\alpha g) = (\alpha f) * g.$$

Hence, $(A, +, *)$ is such that $(A, +)$ is a vector space over \mathbb{C} whereas $(A, +, *)$ is a ring with the additional property (iv). If in addition $*$ is commutative, i.e.

$$(v) \quad f * g = g * f$$

for all $f, g \in A$, then $(A, +, *)$ is called a *commutative algebra*.

If there is a *norm* defined in A which makes A a *normed linear space* and which satisfies the multiplicative inequality

$$(vi) \quad \|f * g\| \leq \|f\| \cdot \|g\|$$

for all $f, g \in A$, then A is called a *normed complex algebra*.

If, in addition, A is a complete metric space with respect to this norm, i.e. if A is a Banach space, then we call A a *Banach algebra*.

An important example of a commutative Banach algebra is $(L^1(\mathbb{R}), +, *)$, where $+$ denotes pointwise addition of functions, and $*$ the convolution of functions as defined before. For more details we refer to GOLDBERG [1] and RUDIN [2].

If A is a commutative algebra and $I \subset A$, then we say that I is an *ideal* in A if

a I is an algebra with respect to the operations in A

b $g * h \in I$ whenever $g \in I$ and $h \in A$.

Clearly A itself is an ideal in A and so is the set consisting of the zero element alone. Any other ideal in A is said to be a proper ideal of A .

An ideal M in A is said to be a *maximal ideal* of A if $M \neq A$ and M is contained in no ideal of A other than M itself or A . The verification of the following three propositions is left to the reader as an exercise.

PROPOSITION 1. If I is an ideal in L^1 , then \bar{I} , the closure of I in L^1 , is also an ideal in L^1 .

PROPOSITION 2. If M is a maximal ideal in L^1 then either M is closed or $M = L^1$.

PROPOSITION 3. If $f \in L^1$, then \bar{T}_f is a closed ideal in L^1 .

We now show that for any $f \in L^1$, \bar{T}_f is the smallest closed ideal in L^1 containing f .

THEOREM 1. If $f \in L^1$ and I is any closed ideal in L^1 containing f , then

$$(1) \quad \bar{T}_f \subset I.$$

PROOF. Let a be any fixed real number. Then for any $n \in \mathbb{N}$

$$(2) \quad \begin{aligned} ((\delta_n)_a * f)(x) &= \int_{-\infty}^{\infty} \delta_n(x+a-t)f(t) dt = \\ &= \int_{-\infty}^{\infty} \delta_n(x-t)f(t+a) dt = \int_{-\infty}^{\infty} \delta_n(x-t)f_a(t) dt \end{aligned}$$

so that

$$(3) \quad (\delta_n)_a * f = \delta_n * f_a.$$

Since I is an ideal and $f \in I$ we thus have $(\delta_n)_a * f \in I$ so that $\delta_n * f_a \in I$. Since I is closed it follows from Theorem 4.1 in Chapter 6 that $f_a \in I$, from which it is clear that $T_f \subset I$. Since I is closed it follows that $\bar{T}_f \subset I$, completing the proof. \square

Now we are able to give the following characterization of closed ideals in L^1 .

THEOREM 2. Let $I \subset L^1$. Then the following statements are equivalent

a I is a closed ideal of L^1

b I is a closed linear subspace of L^1 with the property that if $f \in I$, then every translate of f is also in I .

PROOF. The implication a \Rightarrow b is a direct consequence of the previous theorem.

Now suppose that b holds true and let $g \in I$ and $h \in L^1$. Since I is a linear subspace of L^1 and $g_a \in I$ for all $a \in \mathbb{R}$, it follows that $T_g \subset I$ and since I is closed it also follows that $\overline{T_g}$ is an ideal of L^1 so that $g * h \in \overline{T_g}$ and hence $g * h \in I$. It follows that I is an ideal and since I was given to be closed, the proof is complete. \square

DEFINITION. For each $\lambda \in \mathbb{R}$ we write

$$(4) \quad M_\lambda = \{f \in L^1 \mid \hat{f}(\lambda) = 0\}.$$

THEOREM 3. Every M_λ is a closed maximal ideal in L^1 .

PROOF. (i) M_λ is closed. In order to see this consider the map $\phi: L^1 \rightarrow \mathbb{C}$ defined by

$$(5) \quad \phi(f) := \hat{f}(\lambda).$$

It is easily seen that this map is continuous and since $\phi^{-1}(0) = M_\lambda$, it follows that M_λ is closed.

(ii) M_λ is an ideal in L^1 . Indeed, it is easily verified that M_λ is a subalgebra of L^1 and if $g \in M_\lambda$ and $h \in L^1$, then

$$(6) \quad (g * h)^\wedge(\lambda) = \hat{g}(\lambda) \cdot \hat{h}(\lambda) = 0 \cdot \hat{h}(\lambda) = 0$$

so that $g * h \in M_\lambda$, proving claim (ii).

(iii) M_λ is maximal. In order to see this let M be any ideal in L^1 such that

$$(7) \quad M_\lambda \subset M \quad \text{and} \quad M \neq M_\lambda.$$

Then it suffices to show that $M = L^1$.

Since $M \neq M_\lambda$ there exists a $g \in M$ such that $\hat{g}(\lambda) \neq 0$ so that for any $h \in L^1$ we can define

$$(8) \quad \tilde{h} := h - \frac{\hat{h}(\lambda)}{\hat{g}(\lambda)} \cdot g.$$

Clearly

$$(9) \quad (\tilde{h})^\wedge(\lambda) = 0$$

so that $\tilde{h} \in M$.

Also, by assumption, $g \in M$, so that h (as a linear combination of \tilde{h} and g) belongs to M . It follows that $L^1 \subset M$ and the proof is complete. \square

We leave it as an exercise to the reader to show that $M_{\lambda_1} \neq M_{\lambda_2}$ if $\lambda_1 \neq \lambda_2$.

The next theorem will prepare us to show that the M_λ comprise all maximal ideals of L^1 .

THEOREM 4. *If I is a closed ideal $\neq L^1$, then there exists a $\lambda \in \mathbb{R}$ such that $I \subset M_\lambda$.*

PROOF. We proceed by contradiction. Assume that I is not contained in any M_λ . From this we shall derive that $I = L^1$, which is contradictory to our hypothesis.

Fix any positive integer N . By the assumption that I is not contained in any M_λ , it follows that for each $\lambda \in [-N, N]$ there exists an $f_\lambda \in I$ such that $\hat{f}_\lambda(\lambda) \neq 0$. Let g_λ be defined by

$$(10) \quad g_\lambda(t) := \overline{f_\lambda(-t)}.$$

Then

$$(11) \quad \begin{aligned} \hat{g}_\lambda(x) &= \int_{-\infty}^{\infty} e^{ixt} \overline{f_\lambda(-t)} dt = \\ &= \int_{-\infty}^{\infty} e^{-ixt} \overline{f_\lambda(t)} dt = \overline{\hat{f}_\lambda(x)} \end{aligned}$$

so that

$$(12) \quad \hat{g}_\lambda = \overline{\hat{f}_\lambda}.$$

Defining

$$(13) \quad h_\lambda := g_\lambda * f_\lambda$$

we have

$$(14) \quad h_\lambda \in I \quad \text{and} \quad \hat{h}_\lambda = \hat{g}_\lambda \cdot \hat{f}_\lambda = \overline{\hat{f}_\lambda} \cdot \hat{f}_\lambda = |\hat{f}_\lambda|^2$$

so that

$$(15) \quad \hat{h}_\lambda(x) \geq 0 \quad \text{for all } x \in \mathbb{R} \quad \text{and} \quad \hat{h}_\lambda(\lambda) > 0.$$

Since \hat{h}_λ is continuous, this implies that $\hat{h}_\lambda(x) \geq 0$ for all x in some neighborhood U_λ of λ . The interval $[-N, N]$ can be covered by a finite number of these neighborhoods

$$(16) \quad U_{\lambda_1}, U_{\lambda_2}, \dots, U_{\lambda_n}, \text{ say.}$$

Defining $h \in L^1$ by

$$(17) \quad h := h_{\lambda_1} + h_{\lambda_2} + \dots + h_{\lambda_n}$$

we have

$$(18) \quad h \in I \quad \text{and} \quad \hat{h}(x) > 0 \quad \text{for all} \quad x \in [-N, N].$$

With $\Delta_N = \hat{\delta}_N$ as before we have for some $k \in L^1$

$$(19) \quad \frac{\Delta_N(x)}{\hat{h}(x)} = \Delta_N(x) \cdot \hat{k}(x) \quad \text{for all} \quad x \in \mathbb{R}$$

so that

$$(20) \quad \delta_N = h * k * \delta_N$$

which implies that $\delta_N \in I$. Thus I contains δ_N for all $N \in \mathbb{N}$ and since I is closed it easily follows that $I = L^1$, which is what we wished to show. \square

THEOREM 5. *If M is a closed maximal ideal in L^1 , then $M = M_\lambda$ for some $\lambda \in \mathbb{R}$.*

PROOF. Since M is maximal we have $M \neq L^1$, so that by the previous theorem

$$(21) \quad M \subset M_\lambda \quad \text{for some} \quad \lambda \in \mathbb{R}.$$

Since $M_\lambda \neq L^1$ it follows from the maximality of M that $M = M_\lambda$. \square

Wiener's general Tauberian theorem is a simple consequence of Theorem 4. In order to see this let $f \in L^1$ such that $\hat{f}(x) \neq 0$ for all $x \in \mathbb{R}$. We know that \overline{T}_f is a closed ideal. If $\overline{T}_f \neq L^1$, then, by Theorem 4, there exists a $\lambda \in \mathbb{R}$ such that $\overline{T}_f \subset M_\lambda$ so that $\hat{f}(\lambda) = 0$, contradicting our hypothesis. \square

REFERENCES

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2. RUDIN, W., *Real and Complex Analysis*, McGraw-Hill, 1966.

CHAPTER 10

SOME ANALYTIC REFORMULATIONS

In this chapter we present a collection of analytical consequences of Wiener's general Tauberian theorem.

THEOREM 1. (Wiener) If f is bounded and measurable on \mathbf{R} and $K_1 \in L^1$ is such that $\widehat{K}_1(x) \neq 0$ for all $x \in \mathbf{R}$ and

$$(1) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} f(t) K_1(x-t) dt = 0$$

then also

$$(2) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} f(t) K_2(x-t) dt = 0$$

for every $K_2 \in L^1$.

PROOF. Since $K_1 \in L^1$ and $\widehat{K}_1(x) \neq 0$ for all $x \in \mathbf{R}$ we have

$$(3) \quad \overline{T}_{K_1} = L^1.$$

Fix any $K_2 \in L^1$ and let $\varepsilon > 0$ be given. Choosing $h \in T_{K_1}$ such that

$$(4) \quad \|K_2 - h\|_1 < \frac{\varepsilon}{\|f\|_\infty + 1}$$

we have

$$(5) \quad \left| \int_{-\infty}^{\infty} K_2(x-t)f(t) dt \right| \leq \left| \int_{-\infty}^{\infty} \{K_2(x-t) - h(x-t)\}f(t) dt \right| + \left| \int_{-\infty}^{\infty} h(x-t)f(t) dt \right| \leq \|K_2 - h\|_1 \cdot \|f\|_\infty + \left| \int_{-\infty}^{\infty} h(x-t)f(t) dt \right|.$$

Since $h \in T_{K_1}$ it is clear that

$$(6) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} h(x-t)f(t) dt = 0$$

so that

$$(7) \quad \limsup_{x \rightarrow \infty} \left| \int_{-\infty}^{\infty} K_2(x-t)f(t) dt \right| \leq \frac{\varepsilon}{\|f\|_\infty + 1} \cdot \|f\|_\infty$$

completing the proof. \square

From this theorem we easily derive

THEOREM 2. (Wiener) If f is bounded and measurable on \mathbb{R} and $K_1 \in L^1$ is such that $\hat{K}_1(x) \neq 0$ for all $x \in \mathbb{R}$ and

$$(8) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} K_1(x-t)f(t) dt = L \cdot \int_{-\infty}^{\infty} K_1(t) dt$$

then also

$$(9) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} K_2(x-t)f(t) dt = L \cdot \int_{-\infty}^{\infty} K_2(t) dt$$

for every $K_2 \in L^1$.

PROOF. Observing that for all $x \in \mathbb{R}$

$$(10) \quad \int_{-\infty}^{\infty} K_1(t) dt = \int_{-\infty}^{\infty} K_1(x-t) dt$$

we have

$$(11) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} K_1(x-t)\{f(t) - L\} dt = 0.$$

Hence, the theorem follows from the previous one. \square

We now modify these theorems by an exponential transformation. They then become theorems concerning functions defined on $\mathbb{R}^+ := (0, \infty)$, and it is in this form that they are usually most convenient for applications.

Define

$$(12) \quad F(t) := f(\log t), \quad (t > 0)$$

and

$$(13) \quad k_i(t) := \frac{1}{t} K_i(-\log t), \quad (t > 0; i = 1, 2).$$

Then it is easily verified that for all $x \in \mathbb{R}$ (in particular for $x = 0$)

$$(14) \quad \int_0^{\infty} k_i(t) t^{-ix} dt = \int_{-\infty}^{\infty} K_i(u) e^{iux} du, \quad (i = 1, 2)$$

whereas F is bounded and measurable on \mathbb{R}^+ as soon as f is bounded and measurable on \mathbb{R} . Also, if $K_i \in L^1(\mathbb{R})$ then $k_i \in L^1(\mathbb{R}^+)$ and conversely.

Observing that for all $x > 0$

$$(15) \quad \frac{1}{x} \int_0^{\infty} k_i\left(\frac{t}{x}\right) F(t) dt = \frac{1}{x} \int_0^{\infty} \frac{x}{t} K_i(-\log \frac{t}{x}) f(\log t) dt =$$

$$= \int_{-\infty}^{\infty} K_i((\log x) - u)f(u) du, \quad (i = 1, 2)$$

we easily obtain (replace x by e^x)

THEOREM 3. If F is bounded and measurable on \mathbb{R}^+ and if $k_1 \in L^1(\mathbb{R}^+)$ is such that

$$(16) \quad \int_0^{\infty} t^{-ix} k_1(t) dt \neq 0 \quad \text{for all } x \in \mathbb{R}$$

and

$$(17) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^{\infty} k_1\left(\frac{t}{x}\right) F(t) dt = L \cdot \int_0^{\infty} k_1(t) dt$$

then also

$$(18) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^{\infty} k_2\left(\frac{t}{x}\right) F(t) dt = L \cdot \int_0^{\infty} k_2(t) dt$$

for every $k_2 \in L^1(\mathbb{R}^+)$.

Finally we have

THEOREM 4. If f is bounded and measurable on \mathbb{R}^+ and if $g_1 \in L^1(\mathbb{R}^+)$ is such that

$$(19) \quad \int_0^{\infty} t^{ix} g_1(t) dt \neq 0 \quad \text{for all } x \in \mathbb{R}$$

and

$$(20) \quad \lim_{x \downarrow 0} \frac{1}{x} \int_0^{\infty} g_1\left(\frac{t}{x}\right) f(t) dt = L \cdot \int_0^{\infty} g_1(t) dt$$

then also

$$(21) \quad \lim_{x \downarrow 0} \frac{1}{x} \int_0^{\infty} g_2\left(\frac{t}{x}\right) f(t) dt = L \cdot \int_0^{\infty} g_2(t) dt$$

for every $g_2 \in L^1(\mathbb{R}^+)$.

PROOF. Let F , k_1 and k_2 be as in the previous theorem and define

$$(22) \quad f(t) := F\left(\frac{1}{t}\right), \quad (t > 0)$$

and

$$(23) \quad g_i(t) := \frac{1}{t^2} k_i\left(\frac{1}{t}\right), \quad (t > 0; i = 1, 2).$$

The remaining details of the proof are left to the reader. \square

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CHAPTER 11

PITT'S GENERAL TAUBERIAN THEOREM

1. PITT'S THEOREM FOR SLOWLY OSCILLATING FUNCTIONS ON \mathbb{R}

DEFINITION. A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is called *slowly oscillating* on \mathbb{R} if for every $\epsilon > 0$ there exist numbers $\delta_\epsilon > 0$ and N_ϵ such that $|f(y) - f(x)| \leq \epsilon$ for all x, y satisfying $x \geq N_\epsilon$ and $|y - x| \leq \delta_\epsilon$.

EXAMPLE. If f is differentiable on \mathbb{R} and $|f'(x)| \leq G$ for some fixed G and all sufficiently large $x \in \mathbb{R}$, then f is slowly oscillating on \mathbb{R} .

DEFINITION. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *slowly decreasing* on \mathbb{R} if for every $\epsilon > 0$ there exist numbers $\delta_\epsilon > 0$ and N_ϵ such that $f(y) - f(x) \geq -\epsilon$ for all x, y satisfying $x \geq N_\epsilon$ and $0 \leq y - x \leq \delta_\epsilon$.

EXAMPLES. (i) If f is real valued and slowly oscillating on \mathbb{R} , then f is also slowly decreasing on \mathbb{R} .

(ii) If f is differentiable on \mathbb{R} such that $f'(x) \geq -G$ for some fixed G and all sufficiently large x , then f is slowly decreasing on \mathbb{R} .

(iii) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is monotonically non-decreasing on the interval $x \geq a$ for some a , then f is slowly decreasing on \mathbb{R} .

PROPOSITION 1.1. If f is slowly decreasing on \mathbb{R} and $f(x) \geq \eta > 0$ (for some fixed η) for arbitrarily large $x (= x_n$, say, where $x_n \rightarrow \infty$ as $n \rightarrow \infty$), then $f(x) \geq \frac{\eta}{2}$ in infinitely many (non-overlapping) intervals of fixed length on the positive real axis.

PROOF. In the definition of "slowly decreasing on \mathbb{R} " take $\epsilon = \frac{\eta}{2}$ and corresponding δ and N . Then we have the following implication

$$(1.1) \quad (x \geq N \quad \text{and} \quad 0 \leq y - x \leq \delta) \Rightarrow f(y) - f(x) \geq -\frac{\eta}{2} \Rightarrow \\ \Rightarrow f(y) \geq f(x) - \frac{\eta}{2}.$$

If n_1 is large enough then $x_{n_1} \geq N$ so that $x_{n_1} \leq y \leq x_{n_1} + \delta$ implies that $f(y) \geq \eta - \frac{\eta}{2}$.

Now choose $x_{n_2} > x_{n_1} + 2\delta$ and proceed as before. \square

PROPOSITION 1.2. If f is slowly decreasing on \mathbf{R} and $f(x) \leq -\eta$ (for some fixed positive η) for arbitrarily large values of x ($= x_n$, say) then there exist infinitely many (non-overlapping) intervals on the positive real axis in which $f(x) \leq -\frac{\eta}{2}$.

PROOF. In the definition of "slowly decreasing on \mathbf{R} " take $\epsilon = \frac{\eta}{2}$ and determine corresponding δ and N . Then we have the implication

$$(1.2) \quad (x \geq N \text{ and } 0 \leq y-x \leq \delta) \Rightarrow f(y) - f(x) \geq -\frac{\eta}{2}.$$

Take n such that $x_n - \delta \geq N$ and let $y = x_n$. Then for all $x \in [x_n - \delta, x_n]$ we have $x \geq x_n - \delta \geq N$ and $0 \leq y-x = x_n - x \leq \delta$ and hence $f(x_n) - f(x) \geq -\frac{\eta}{2}$ or

$$(1.3) \quad f(x) \leq \frac{\eta}{2} + f(x_n) \leq \frac{\eta}{2} - \eta = -\frac{\eta}{2}, \quad (x \in [x_n - \delta, x_n]).$$

From here on proceed as before. \square

THEOREM 1.1. If f is bounded and slowly decreasing on \mathbf{R} and if for every $\lambda > 0$

$$(1.4) \quad \lim_{x \rightarrow \infty} (\delta_\lambda * f)(x) = A, \quad (\text{where } \delta_\lambda \text{ is as on page 46})$$

then

$$(1.5) \quad \lim_{x \rightarrow \infty} f(x) = A.$$

PROOF. Since

$$(1.6) \quad \begin{aligned} \int_{-\infty}^{\infty} \delta_\lambda(x-t) dt &= \int_{-\infty}^{\infty} \delta_\lambda(t) dt = \\ &= \int_{-\infty}^{\infty} \lambda \delta(\lambda t) dt = \int_{-\infty}^{\infty} \delta(t) dt = 1 \end{aligned}$$

it is no essential restriction to assume that $A = 0$. Proceeding by contradiction we assume that $f(x)$ does not tend to 0 as $x \rightarrow \infty$. Then there exists a positive η such that $f(x) \geq \eta$ or $f(x) \leq -\eta$ for arbitrarily large values of x . Let us assume the first alternative (leaving the other alternative to the reader). By the previous proposition there exist infinitely many non-overlapping intervals $[x_n - \xi, x_n + \xi]$ on the positive real axis in which $f(x) \geq \frac{\eta}{2}$. Now observe that for all $n \in \mathbf{N}$

$$(1.7) \quad \int_{-\infty}^{\infty} \delta_\lambda(x_n - t) f(t) dt = \int_{x_n - \xi}^{x_n + \xi} \delta_\lambda(x_n - t) f(t) dt +$$

$$\begin{aligned}
& + \left\{ \int_{-\infty}^{x_n - \xi} + \int_{x_n + \xi}^{\infty} \right\} \delta_\lambda(x_n - t) f(t) dt \geq \\
& \geq \frac{\eta}{2} \int_{x_n - \xi}^{x_n + \xi} \delta_\lambda(x_n - t) dt - \|f\|_\infty \left\{ \int_{-\infty}^{x_n - \xi} + \int_{x_n + \xi}^{\infty} \right\} \delta_\lambda(x_n - t) dt = \\
& = \eta \int_0^\xi \delta_\lambda(t) dt - 2\|f\|_\infty \int_\xi^\infty \delta_\lambda(t) dt = \\
& = \eta \int_0^{\lambda\xi} \delta(t) dt - 2\|f\|_\infty \int_{\lambda\xi}^\infty \delta(t) dt.
\end{aligned}$$

Since

$$\begin{aligned}
(1.8) \quad & \lim_{\lambda \rightarrow \infty} \left\{ \eta \int_0^{\lambda\xi} \delta(t) dt - 2\|f\|_\infty \int_{\lambda\xi}^\infty \delta(t) dt \right\} = \\
& = \eta \int_0^\infty \delta(t) dt = \frac{\eta}{2}
\end{aligned}$$

we may determine λ_0 such that for all $n \in \mathbb{N}$

$$(1.9) \quad \int_{-\infty}^\infty \delta_{\lambda_0}(x_n - t) f(t) dt \geq \frac{\eta}{4}.$$

Letting $n \rightarrow \infty$ we arrive at a contradiction, proving the theorem. \square

THEOREM 1.2. (Pitt) *Let $a : \mathbb{R} \rightarrow \mathbb{R}$ be bounded and slowly decreasing on \mathbb{R} . If $g \in L^1$ and $\hat{g}(x) \neq 0$ for all $x \in \mathbb{R}$, then*

$$(1.10) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^\infty g(x-t) a(t) dt = A \int_{-\infty}^\infty g(t) dt$$

implies that

$$(1.11) \quad \lim_{x \rightarrow \infty} a(x) = A.$$

PROOF. By the previous theorem we only need to show that

$$(1.12) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^\infty \delta_\lambda(x-t) a(t) dt = A \left(= A \int_{-\infty}^\infty \delta_\lambda(t) dt \right).$$

However, since $\delta_\lambda \in L^1$ and $\hat{g}(x) \neq 0$ for all $x \in \mathbb{R}$ the above limit relation is an immediate consequence of Wiener's general Tauberian theorem, completing the proof. \square

We now want to show how Wiener's theorem (page 69) can be obtained from Pitt's theorem.

Let f be bounded and measurable on \mathbb{R} and let $g_1 \in L^1(\mathbb{R})$ be such that $\hat{g}_1(x) \neq 0$, for all $x \in \mathbb{R}$. Fix any $g_2 \in L^1(\mathbb{R})$ and define

$$(1.13) \quad a(x) := \int_{-\infty}^{\infty} g_2(x-t) f(t) dt, \quad (x \in \mathbb{R}).$$

Then a is continuous (and hence measurable) and slowly oscillating on \mathbb{R} . In order to see this we observe that

$$(1.14) \quad \begin{aligned} |a(y)-a(x)| &= \left| \int_{-\infty}^{\infty} g_2(y-t)f(t) dt - \int_{-\infty}^{\infty} g_2(x-t)f(t) dt \right| = \\ &= \left| \int_{-\infty}^{\infty} \{g_2(y-t) - g_2(x-t)\} f(t) dt \right| \leq \\ &\leq \|f\|_{\infty} \cdot \int_{-\infty}^{\infty} |g_2(y-t) - g_2(x-t)| dt = \\ &= \|f\|_{\infty} \cdot \int_{-\infty}^{\infty} |g_2(y-x+u) - g_2(u)| du. \end{aligned}$$

Since $g_2 \in L^1(\mathbb{R})$ we have

$$(1.15) \quad \lim_{y-x \rightarrow 0} \int_{-\infty}^{\infty} |g_2(y-x+u) - g_2(u)| du = 0$$

so that a is slowly oscillating on \mathbb{R} .

Now assume that

$$(1.16) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} g_1(x-t)f(t) dt = 0 \quad (\text{or} = A \cdot \int_{-\infty}^{\infty} g_1(t) dt).$$

We will show that then

$$(1.17) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} g_1(x-t)a(t) dt = 0 \quad (\text{or} = A \cdot \int_{-\infty}^{\infty} g_2(t)dt \cdot \int_{-\infty}^{\infty} g_1(t)dt).$$

Observe that (using Fubini's theorem)

$$(1.18) \quad \begin{aligned} \int_{-\infty}^{\infty} g_1(x-t) a(t) dt &= \int_{-\infty}^{\infty} g_1(x-t) \int_{-\infty}^{\infty} g_2(t-u) f(u) du dt = \\ &= \int_{-\infty}^{\infty} g_1(x-t) \int_{-\infty}^{\infty} g_2(u) f(t-u) du dt = \\ &= \int_{-\infty}^{\infty} g_2(u) \int_{-\infty}^{\infty} g_1(x-t) f(t-u) dt du = \\ &= \int_{-\infty}^{\infty} g_2(u) \int_{-\infty}^{\infty} g_1(x-t-u) f(t) dt du. \end{aligned}$$

Since

$$(1.19) \quad \left| g_2(u) \int_{-\infty}^{\infty} g_1(x-t-u) f(t) dt \right| \leq \\ \leq |g_2(u)| \cdot \|g_1\|_1 \cdot \|f\|_{\infty}, \quad g_2 \in L^1(\mathbb{R})$$

and

$$(1.20) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} g_1(x-t-u) f(t) dt = 0 \quad (\text{or} = A \cdot \int_{-\infty}^{\infty} g_1(t) dt)$$

we find by LDCT

$$(1.21) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} g_1(x-t) a(t) dt = 0 \quad (\text{or} = A \cdot \int_{-\infty}^{\infty} g_2(t) dt \cdot \int_{-\infty}^{\infty} g_1(t) dt)$$

and Pitt's theorem yields

$$(1.22) \quad \lim_{x \rightarrow \infty} a(x) = \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} g_2(x-t) f(t) dt = 0 \quad (\text{or} = A \cdot \int_{-\infty}^{\infty} g_2(t) dt)$$

proving Wiener's Tauberian theorem. \square

2. PITT'S THEOREM FOR SLOWLY OSCILLATING FUNCTIONS ON \mathbb{R}^+

DEFINITION. A function $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ is called *slowly oscillating* on \mathbb{R}^+ if for every $\epsilon > 0$ there exist numbers $\delta_{\epsilon} > 0$ and N_{ϵ} such that $|f(y) - f(x)| \leq \epsilon$ for all x, y satisfying $x \geq N_{\epsilon}$ and $|\frac{y}{x} - 1| \leq \delta_{\epsilon}$.

DEFINITION. A function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is called *slowly decreasing* on \mathbb{R}^+ if for every $\epsilon > 0$ there exist numbers $\delta_{\epsilon} > 0$ and N_{ϵ} such that $f(y) - f(x) \geq -\epsilon$ for all x, y satisfying $x \geq N_{\epsilon}$ and $1 \leq \frac{y}{x} - 1 \leq 1 + \delta_{\epsilon}$.

Clearly any real f that is slowly oscillating on \mathbb{R}^+ is slowly decreasing on \mathbb{R}^+ .

Pitt's theorem may be translated into the following

THEOREM 2.1. Let a be bounded and slowly decreasing on \mathbb{R}^+ .

If $g \in L^1(\mathbb{R}^+)$ and $\int_0^{\infty} t^{ix} g(t) dt \neq 0$, for all $x \in \mathbb{R}$, then

$$(2.1) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^{\infty} g\left(\frac{t}{x}\right) a(t) dt = A \int_0^{\infty} g(t) dt$$

implies

$$(2.2) \quad \lim_{x \rightarrow \infty} a(x) = A.$$

The proof is left to the reader. \square

REMARK. In order to speak the same sort of language as in Chapter 4 we note that for $x > 0$

$$(2.3) \quad \frac{1}{x} \int_0^{\infty} g\left(\frac{t}{x}\right) a(t) dt = \int_0^{\infty} g(v) a(xv) dv.$$

3. SOME APPLICATIONS

APPLICATION 1. Let F be of bounded variation over $[0, T]$, for all $T > 0$. Also let F be bounded on $[0, \infty)$ and assume that F is slowly oscillating (or slowly decreasing) on \mathbb{R}^+ . Furthermore assume that

$$(3.1) \quad \lim_{s \downarrow 0} \int_0^{\infty} e^{-st} dF(t) = L.$$

Then

$$(3.2) \quad \lim_{t \rightarrow \infty} F(t) = L + F(0).$$

PROOF. Observe that

$$(3.3) \quad \begin{aligned} \int_0^{\infty} e^{-st} dF(t) &= e^{-st} F(t) \Big|_0^{\infty} - \int_0^{\infty} F(t) de^{-st} = \\ &= -F(0) + s \int_0^{\infty} F(t) e^{-st} dt, \quad (s > 0). \end{aligned}$$

Hence

$$(3.4) \quad s \int_0^{\infty} e^{-st} F(t) dt = F(0) + \int_0^{\infty} e^{-st} dF(t)$$

so that

$$(3.5) \quad \lim_{s \downarrow 0} s \int_0^{\infty} e^{-st} F(t) dt = F(0) + L$$

or, equivalently, (setting $s = \frac{1}{x}$)

$$(3.6) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^{\infty} e^{-\frac{t}{x}} F(t) dt = (F(0) + L) \cdot \int_0^{\infty} e^{-t} dt.$$

Pitt's Theorem 2.1 applies(!) and we obtain (3.2). \square

APPLICATION 2. Let F be bounded and slowly decreasing on \mathbb{R}^+ .

If in addition

$$(3.7) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x F(t) dt = L$$

then

$$(3.8) \quad \lim_{t \rightarrow \infty} F(t) = L.$$

PROOF. Define $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ as follows

$$g(x) = \begin{cases} 1, & (0 < x \leq 1) \\ 0, & (x > 1). \end{cases}$$

Then $g \in L^1(\mathbb{R}^+)$, $\int_0^\infty g(t) dt = 1$ and $\int_0^\infty t^{ix} g(t) dt = \int_0^1 t^{ix} dt = \frac{1}{1+ix}$
so that $\int_0^\infty t^{ix} g(t) dt \neq 0$ for all $x \in \mathbb{R}$.

Now observe that (3.7) may be written as

$$(3.10) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^\infty g\left(\frac{t}{x}\right) F(t) dt = L \cdot \int_0^\infty g(t) dt.$$

Pitt's theorem applies and we obtain (3.8). \square

LITERATURE.

(See page 72)

A RELATED TOPIC: CLOSED SYSTEMS

Let V be a vector space over the complex field \mathbb{C} and let G be a subset of V . By the *span* of G we mean the set of all finite linear combinations

$$(1) \quad \sum_{k=1}^n \lambda_k g_k, \quad (n \in \mathbb{N}; \lambda_k \in \mathbb{C}; g_k \in G).$$

This set will be denoted by $H(G)$.

If in addition V is a topological vector space then G is called a *fundamental* (or *closed*) *system* in V if $H(G)$ is dense in V , i.e. $\overline{H(G)} = V$. Using this terminology we may formulate Wiener's theorem (2.4 in Chapter 9) as follows: If $f \in L^1(\mathbb{R})$ then T_f is a fundamental system in $L^1(\mathbb{R})$ if and only if $\hat{f}(t) \neq 0, \forall t \in \mathbb{R}$.

From this we derive the following

THEOREM 1. Let $\phi \in L^1(\mathbb{R}^+)$. Then the system $\{\phi(\lambda x)\}_{\lambda \in \mathbb{R}^+}$ is a closed system in $L^1(\mathbb{R}^+)$ if and only if $\int_0^{\infty} \phi(x)x^{it} dx \neq 0, \forall t \in \mathbb{R}$.

PROOF. For any $f \in L^1(\mathbb{R}^+)$ define

$$(2) \quad f^*(u) := e^u f(e^u), \quad (u \in \mathbb{R}).$$

Then it is easily seen that $f^* \in L^1(\mathbb{R})$. Also

$$(3) \quad \begin{aligned} \hat{\phi}^*(t) &= \int_{-\infty}^{\infty} \phi^*(u) e^{iut} du = \\ &= \int_{-\infty}^{\infty} e^u \phi(e^u) e^{iut} du = \int_0^{\infty} \phi(x)x^{it} dx \neq 0, \forall t \in \mathbb{R}. \end{aligned}$$

By Wiener's theorem we may choose constants $c_n \in \mathbb{C}$ and $h_n \in \mathbb{R}$ such that

$$(4) \quad \|f^*(u) - \sum_{n=1}^N c_n \phi^*(u+h_n)\|_1 < \varepsilon.$$

The left-hand side of this inequality may be rewritten as

$$(5) \quad \begin{aligned} \int_{-\infty}^{\infty} |f^*(u) - \sum_{n=1}^N c_n \phi^*(u+h_n)| du = \\ = \int_{-\infty}^{\infty} |e^u f(e^u) - \sum_{n=1}^N c_n e^{u+h_n} \phi(e^{u+h_n})| du = \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\infty} \left| f(x) - \sum_{n=1}^N c_n \lambda_n \phi(\lambda_n x) \right| dx = \\
&= \int_0^{\infty} \left| f(x) - \sum_{n=1}^N \beta_n \phi(\lambda_n x) \right| dx
\end{aligned}$$

where $\beta_n = c_n \lambda_n = c_n e^{\ln \lambda_n} \in \mathbb{C}$ and $\lambda_n > 0$.

Hence

$$(6) \quad \left\| f(x) - \sum_{n=1}^N \beta_n \phi(\lambda_n x) \right\|_{L^1(\mathbb{R}^+)} < \varepsilon$$

and the proof is complete. \square

We leave it to the reader to show that if the system $\{\phi(\lambda x)\}_{\lambda \in \mathbb{R}^+}$ is fundamental in $L^1(\mathbb{R}^+)$, then $\int_0^{\infty} \phi(x) x^{it} dx \neq 0$, $\forall t \in \mathbb{R}$.

EXAMPLE. Let $\phi(x) = e^{-x}$, $x > 0$. Then $\phi \in L^1(\mathbb{R}^+)$ and

$$(7) \quad \int_0^{\infty} \phi(x) x^{it} dx = \int_0^{\infty} e^{-x} x^{it} dx = \Gamma(1+it).$$

Since the Gamma-function has no zeros at all the system $\{e^{-\lambda x}\}_{\lambda > 0}$ is fundamental in $L^1(\mathbb{R}^+)$.

THEOREM 2. Let $\phi \in L^1(0,1)$. Then the system $\{x^{\lambda-1} \phi(x^\lambda)\}_{\lambda > 0}$ is fundamental in $L^1(0,1)$ if and only if

$$(8) \quad \int_0^1 \phi(x) \left(\log \frac{1}{x}\right)^{it} dx \neq 0, \quad \forall t \in \mathbb{R}.$$

PROOF. For any $f \in L^1(0,1)$ define

$$(9) \quad f^*(u) = e^{-u} f(e^{-u}), \quad (u > 0).$$

Then $f^* \in L^1(\mathbb{R}^+)$ and

$$(10) \quad \int_0^{\infty} \phi^*(u) u^{it} du = \int_0^1 \phi(x) \left(\log \frac{1}{x}\right)^{it} dx \neq 0, \quad \forall t \in \mathbb{R}.$$

The completion of the proof is left to the reader (compare the proof of Theorem 1). \square

EXAMPLE 1. Let $\phi(x) = 1$, $(0 < x < 1)$. Then

$$\begin{aligned}
(11) \quad \int_0^1 \phi(x) \left(\log \frac{1}{x}\right)^{it} dx &= \int_0^1 \left(\log \frac{1}{x}\right)^{it} dx = \\
&= \int_0^{\infty} u^{it} e^{-u} du = \Gamma(1+it) \neq 0, \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Thus the system $\{x^{\lambda-1}\}_{\lambda>0}$ is fundamental in $L^1(0,1)$.

(Compare this result with the Stone-Weierstrass theorem.)

EXAMPLE 2. Let N be a fixed positive integer and let $\phi(x) = \frac{1-x^N}{1-x}$. Then $\phi \in L^1(0,1)$ and

$$\begin{aligned}
 (12) \quad \int_0^1 \phi(x) \left(\log \frac{1}{x}\right)^{it} dx &= \int_0^{\infty} \frac{1-e^{-Nu}}{1-e^{-u}} u^{it} e^{-u} du = \\
 &= \int_0^{\infty} (e^{-u} + e^{-2u} + \dots + e^{-Nu}) u^{it} du = \\
 &= \sum_{n=1}^N \int_0^{\infty} e^{-nu} u^{it} du = \sum_{n=1}^N \int_0^{\infty} e^{-x} \left(\frac{x}{n}\right)^{it} \frac{dx}{n} = \\
 &= \Gamma(1+it) \sum_{n=1}^N \frac{1}{n^{1+it}}.
 \end{aligned}$$

Since the Gamma-function has no zeros this leads us to the question whether $\sum_{n=1}^N n^{-1-it}$ has any real zeros. For $N \leq 10$ it is known that this function does not vanish on \mathbb{R} (see [2]). For large values of N the answer is not known, although we conjecture that for all $N \in \mathbb{N}$ the above function does not vanish on the real line.

EXAMPLE 3. Let $\phi(x) = \frac{1}{(1+x)^2}$, ($0 < x < 1$). Then we have

$$\begin{aligned}
 (13) \quad \int_0^1 \phi(x) \left(\log \frac{1}{x}\right)^{it} dx &= \int_0^{\infty} \frac{e^{-x}}{(1+e^{-x})^2} x^{it} dx = \\
 &= \lim_{\sigma \rightarrow 0} \int_0^{\infty} \frac{e^{-x}}{(1+e^{-x})^2} x^{\sigma+it} dx.
 \end{aligned}$$

For $\sigma > 0$ we have ($s = \sigma+it$)

$$\begin{aligned}
 (14) \quad \int_0^{\infty} \frac{e^{-x}}{(1+e^{-x})^2} x^s dx &= \int_0^{\infty} x^s d\frac{1}{1+e^{-x}} = \\
 &= \int_0^{\infty} x^s d\frac{e^x}{1+e^x} = - \int_0^{\infty} x^s d\frac{1}{1+e^x} = \\
 &= s \int_0^{\infty} \frac{x^{s-1}}{1+e^x} dx = s\Gamma(s)\eta(s) = s\Gamma(s)(1-2^{1-s})\zeta(s)
 \end{aligned}$$

where

$$\begin{aligned}
 (15) \quad \eta(s) &:= \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \quad \text{and} \quad \zeta(s) := \sum_{n=1}^{\infty} n^{-s}. \\
 &(\sigma > 0) \qquad \qquad \qquad (\sigma > 1)
 \end{aligned}$$

It follows that

$$(16) \quad \int_0^1 \phi(x) \left(\log \frac{1}{x}\right)^{it} dx = \Gamma(1+it) (1-2^{1-it}) \zeta(it).$$

We thus have to investigate whether $\zeta(it) \neq 0$, for all $t \in \mathbb{R}$. It is well known that $\zeta(1+it) \neq 0$, for all $t \in \mathbb{R}$ and that the zeta-function satisfies the functional equation

$$(17) \quad \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1}{2}+s} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

(see TITCHMARSH [4; p. 22]). From this it is easily seen that $\zeta(it) \neq 0$, for all $t \in \mathbb{R}$.

Conclusion: The system $\left\{ \frac{x^{\lambda-1}}{(1+x^\lambda)^2} \right\}_{\lambda>0}$ is fundamental in $L^1(0,1)$.

In the remaining part of this chapter we will use this result to prove that the system $\left\{ \frac{x^n}{1+x^n} \right\}_{n=0}^\infty$ is fundamental in $C[0,1]$.

In order to do so we first prove the following

PROPOSITION 1. Suppose that $\{\phi_\alpha\}_{\alpha \in A}$ is a collection of continuous functions on $[a,b]$ such that for every $\alpha \in A (= \text{index set})$ we have

$$(18) \quad \phi'_\alpha \in L^1[a,b].$$

If $\{\phi_\alpha\}_{\alpha \in A}$ is a fundamental system in $L^1[a,b]$, then $\{1\} \cup \{\phi_\alpha\}_{\alpha \in A}$ is a fundamental system in $C[a,b]$ (equipped with the topology of uniform convergence).

PROOF. Let $f \in C[a,b]$ and $\epsilon > 0$ be given. According to the Stone-Weierstrass theorem there exists a polynomial $P(x)$ such that

$$(19) \quad \max_{x \in [a,b]} |f(x) - P(x)| < \frac{\epsilon}{2}.$$

Clearly $P'(x)$ belongs to $L^1[a,b]$ so that we may choose coefficients $b_n \in \mathbb{C}$ such that

$$(20) \quad \|P' - \sum_{n=1}^N b_n \phi'_n\|_1 < \frac{\epsilon}{2}.$$

For all $x \in [a,b]$ we thus have

$$(21) \quad \begin{aligned} & \left| \left\{ P(x) - \sum_{n=1}^N b_n \phi_n(x) \right\} - \left\{ P(a) - \sum_{n=1}^N b_n \phi_n(a) \right\} \right| = \\ & = \left| \int_a^x \left\{ P'(t) - \sum_{n=1}^N b_n \phi'_n(t) \right\} dt \right| \leq \end{aligned}$$

$$\begin{aligned} &\leq \int_a^b |P'(t) - \sum_{n=1}^N b_n \phi_n'(t)| dt = \\ &= \|P' - \sum_{n=1}^N b_n \phi_n'\|_1 < \frac{\varepsilon}{2}. \end{aligned}$$

Hence

$$\begin{aligned} (22) \quad &|f(x) - \sum_{n=1}^N b_n \phi_n(x) - \{P(a) - \sum_{n=1}^N b_n \phi_n(a)\}| < \\ &< |f(x) - P(x)| + \frac{\varepsilon}{2} < \varepsilon, \quad \forall x \in [a, b] \end{aligned}$$

and it follows that $\{1\} \cup \{\phi_\alpha\}_{\alpha \in A}$ is fundamental in $C[a, b]$. \square

Hence, in order to reach our goal it suffices to prove that the system

$$(23) \quad \left\{ \frac{d}{dx} \frac{x^n}{1+x^n} \right\}_{n=1}^{\infty} = \left\{ \frac{nx^{n-1}}{(1+x^n)^2} \right\}_{n=1}^{\infty}$$

is fundamental in $L^1[0, 1]$.

In order to prove this we will use the following

PROPOSITION 2. *Let G be a subset of the normed linear space V (over \mathbb{E}). Then G is a fundamental system in V if and only if every continuous linear functional on V which vanishes on G , vanishes on all of V .*

PROOF. Assume G to be a fundamental system in V and let ϕ be any continuous linear functional on V which vanishes on G . Then ϕ vanishes on $H(G)$ by linearity and on $\overline{H(G)}$ by continuity. The remaining part of the proof is a simple consequence of the Hahn-Banach theorem for normed linear spaces (see RUDIN [3; p. 108]). \square

Hence, in order to reach our goal we take any continuous linear functional ϕ on $L^1[0, 1]$ such that for every positive integer n , $\phi(x^{n-1}/(1+x^n)^2) = 0$ and then show that such a functional must vanish identically on $L^1[0, 1]$.

It stands to reason that we now want to know all continuous linear functionals on $L^1[0, 1]$. Therefore we state the following

PROPOSITION 3. *For every continuous linear functional ϕ on $L^1[0, 1]$ there exists a "unique" bounded measurable function $\tilde{\phi}$ on $[0, 1]$*

such that

$$(24) \quad \phi(f) = \int_0^1 f(x) \tilde{\phi}(x) dx, \quad \forall f \in L^1[0,1].$$

PROOF. See RUDIN [3; p. 128]. \square

So we take any bounded measurable function $\tilde{\phi}$ on $[0,1]$ such that for all $n \in \mathbb{N}$

$$(25) \quad \int_0^1 \frac{x^{n-1}}{(1+x^n)^2} \tilde{\phi}(x) dx = 0$$

and from this we will derive that then $\tilde{\phi}$ must be a null-function. We consider

$$(26) \quad H(s) = \int_0^1 \frac{x^{s-1}}{(1+x^s)^2} \tilde{\phi}(x) dx, \quad (\operatorname{Re} s > 0)$$

and will show that there exist positive constants K and δ such that

$$(27) \quad |H(s)| \leq K \cdot |s|, \quad \operatorname{Re} s \geq \delta.$$

Writing $A = \|\tilde{\phi}\|_{\infty}$, we have

$$(28) \quad |H(s)| \leq A \cdot \int_0^1 \left| \frac{x^{s-1}}{(1+x^s)^2} \right| dx = \\ = \int_0^{\infty} \left| \frac{e^{-su}}{(1+e^{-su})^2} \right| du = \\ = \left\{ \int_0^{\frac{1}{|s|}} + \int_{\frac{1}{|s|}}^{\infty} \right\} \left| \frac{e^{-su}}{(1+e^{-su})^2} \right| du = I_1 + I_2, \text{ say.}$$

Setting $s = \sigma + it$ we have

$$(29) \quad e^{-su} = e^{-\sigma u - itu}$$

so that, for $0 \leq u \leq \frac{1}{|s|}$,

$$(30) \quad |\arg e^{-su}| = |-tu| \leq u|s| \leq 1$$

and hence (draw a picture)

$$(31) \quad |1 + e^{-su}| \geq 1.$$

Consequently we have

$$(32) \quad I_1 \leq \int_0^{\frac{1}{|s|}} e^{-\sigma u} du \leq \int_0^{\infty} e^{-\sigma u} du = \frac{1}{\sigma}.$$

For I_2 we have the following estimate

$$\begin{aligned}
 (33) \quad I_2 &= \int_0^{\infty} \frac{|e^{-su}|}{|s| (1+e^{-su})^2} du \leq \\
 &\leq \int_0^{\infty} \frac{e^{-\sigma u}}{|s| (1-e^{-\sigma u})^2} du = \frac{-1}{1-e^{-\sigma}} \Big|_0^{\infty} \frac{1}{|s|} = \\
 &= -1 + \frac{1}{1-e^{-\sigma/|s|}} \leq \frac{1}{1-e^{-\sigma/|s|}} = \\
 &= \frac{\sigma/|s|}{1-e^{-\sigma/|s|}} \cdot \frac{|s|}{\sigma} \leq G \frac{|s|}{\sigma}.
 \end{aligned}$$

Hence, for $\sigma \geq \delta > 0$,

$$(34) \quad I_1 + I_2 \leq \frac{1}{\sigma} + \frac{G|s|}{\sigma} \leq K \cdot |s|. \quad \square$$

Now we utilize the following

PROPOSITION 4. If $f(z)$ is regular on $\sigma \geq 0$ and $f(z) = O(e^{k|z|})$ on $\sigma \geq 0$ for some $k < \pi$ and if $f(z) = 0$ for $z = 0, 1, 2, 3, \dots$, then $f(z) = 0$ identically.

PROOF. See TITCHMARSH [4; p. 168]. \square

In this proposition we take

$$(35) \quad f(z) = \frac{H(z+1)}{z+1}, \quad (\operatorname{Re} z \geq 0).$$

Then $f(z)$ is bounded on $\operatorname{Re} z \geq 0$ so that we may take $k = 0$. Furthermore we have by assumption

$$(36) \quad f(n) = \frac{H(n+1)}{n+1} = \frac{1}{n+1} \int_0^1 \frac{x^n}{(1+x^{n+1})^2} \tilde{\phi}(x) dx = 0$$

and it follows that $H(z) = 0$ identically.

In particular we find that

$$(37) \quad \int_0^1 \frac{x^{\lambda-1}}{(1+x^\lambda)^2} \tilde{\phi}(x) dx = 0, \quad \forall \lambda > 0.$$

We already know that the system $\left\{ \frac{x^{\lambda-1}}{(1+x^\lambda)^2} \right\}_{\lambda > 0}$ is fundamental in $L^1[0,1]$.

According to Propositions 2 and 3 we thus obtain that $\tilde{\phi}$ is a null-function, so that the system $\left\{\frac{x^{n-1}}{(1+x^n)^2}\right\}_{n=1}^{\infty}$ is fundamental in $L^1[0,1]$.

Conclusion: The system $\left\{\frac{x^n}{1+x^n}\right\}_{n=0}^{\infty}$ is fundamental in $C[0,1]$.

REMARK. Example 3 has been taken from KOREVAAR [1].

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CHAPTER 13

IKEHARA'S THEOREM

0. INTRODUCTION

Let $F(x) := Ae^x$, ($A \neq 0$; $x > 0$). Then the Laplace transform of F is

$$(0.1) \quad \phi(s) := \int_0^{\infty} e^{-sx} Ae^x dx = \frac{A}{s-1}, \quad (s = \sigma + it \in \mathbb{C}; \sigma > 1).$$

Note that ϕ is regular (analytic, holomorphic) on \mathbb{C} except at $s = 1$ where it has a simple pole with residue A .

Now let G be a measurable function such that $G(x)e^{-x}$ is bounded on \mathbb{R}^+ and $\lim_{x \rightarrow \infty} G(x)e^{-x} = A$. Then the Laplace transform of G is regular on $\sigma > 1$. Since F and G share a certain property, namely $F(x) \sim Ae^x$ and $G(x) \sim Ae^x$, as $x \rightarrow \infty$, one may expect that also the Laplace transforms of F and G have some properties in common.

THEOREM 0.1. *If $F(x) \sim Ae^x$, ($x \rightarrow \infty$), then the Laplace transform ϕ of F has no poles on $\sigma = 1$, except perhaps at $s = 1$.*

PROOF. For $\sigma > 1$ we have

$$(0.2) \quad \begin{aligned} \phi(s) - \frac{A}{s-1} &= \int_0^{\infty} e^{-sx} F(x) dx - \int_0^{\infty} Ae^{-(s-1)x} dx = \\ &= \int_0^{\infty} e^{-(s-1)x} \left\{ \frac{F(x)}{e^x} - A \right\} dx. \end{aligned}$$

Take $s = 1 + \delta + y_0 i$, $\delta > 0$ and $s_0 = 1 + y_0 i$. Then

$$(0.3) \quad \begin{aligned} |(s-s_0) \left\{ \phi(s) - \frac{A}{s-1} \right\}| &= \delta \left| \int_0^{\infty} e^{-(\delta+y_0 i)x} \left\{ \frac{F(x)}{e^x} - A \right\} dx \right| \leq \\ &\leq \delta \int_0^{\infty} e^{-\delta x} \left| \frac{F(x)}{e^x} - A \right| dx = \delta \left\{ \int_0^T + \int_T^{\infty} \right\} e^{-\delta x} \left| \frac{F(x)}{e^x} - A \right| dx \leq \\ &\leq \delta \int_0^T \left| \frac{F(x)}{e^x} - A \right| dx + \delta \cdot \varepsilon_T \cdot \int_T^{\infty} e^{-\delta x} dx \leq \\ &\leq \delta \int_0^T \left| \frac{F(x)}{e^x} - A \right| dx + \varepsilon_T \end{aligned}$$

where

$$(0.4) \quad \varepsilon_T := \sup_{x \geq T} \left| \frac{F(x)}{e^x} - A \right|, \quad (T > 0).$$

Hence

$$(0.5) \quad \limsup_{\delta \downarrow 0} |(s-s_0)\{\phi(s) - \frac{A}{s-1}\}| \leq \epsilon_T$$

and since $\lim_{T \rightarrow \infty} \epsilon_T = 0$ we obtain

$$(0.6) \quad \lim_{\delta \downarrow 0} (s-s_0)\{\phi(s) - \frac{A}{s-1}\} = 0.$$

If $y_0 \neq 0$ this is equivalent to

$$(0.7) \quad \lim_{\delta \downarrow 0} (s-s_0)\phi(s) = 0$$

and from this it is clear that in case ϕ has an analytic continuation up to $\sigma \geq 1$, $s \neq 1$, it can have no poles on $\sigma = 1$, $s \neq 1$. \square

The (Tauberian) theorem in the next section contains some conditions under which the converse of the previous (Abelian) theorem holds true.

1. IKEHARA'S THEOREM

THEOREM 1.1. (1931, IKEHARA [3]) *Suppose that $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is non-decreasing and non-negative and that the Laplace transform*

$$(1.1) \quad \phi(s) = \int_0^{\infty} e^{-sx} F(x) dx$$

converges for $\sigma > 1$.

If

$$(1.2) \quad \Delta(s) := \phi(s) - \frac{A}{s-1} = \phi(1+\alpha+\beta i) - \frac{A}{\alpha+\beta i}, \quad (\alpha > 0)$$

tends uniformly to a limit function $r(\beta)$ if $\alpha \downarrow 0$ and $-\lambda \leq \beta \leq \lambda$ (where λ may be taken as large as we please), then

$$(1.3) \quad \lim_{x \rightarrow \infty} \frac{F(x)}{e^x} = A.$$

REMARK. If $\phi(s)$ has an analytic continuation up to $\sigma \geq 1$, $s \neq 1$, having a simple pole at $s = 1$ with residue A , then the conditions in Ikehara's theorem are certainly satisfied.

PROOF. Writing $L(x) := e^{-x}F(x)$ we have for $\alpha > 0$

$$(1.4) \quad \Delta(1+\alpha+\beta i) = \int_0^{\infty} e^{-\beta ix} e^{-\alpha x} L(x) dx - \int_0^{\infty} e^{-\beta ix} e^{-\alpha x} A dx.$$

Let $H(\beta)$ be complex valued and continuous on $[-2\lambda, 2\lambda]$.

Then the integral

$$(1.5) \quad \int_{-2\lambda}^{2\lambda} H(\beta) \left\{ \int_0^{\infty} e^{-\beta ix} e^{-\alpha x} L(x) dx \right\} d\beta$$

exists and equals (see TITCHMARSH [4; p. 392])

$$(1.6) \quad \int_0^{\infty} e^{-\alpha x} L(x) \left\{ \int_{-2\lambda}^{2\lambda} H(\beta) e^{-\beta ix} d\beta \right\} dx.$$

In particular this holds for

$$(1.7) \quad H(\beta) = \frac{1}{\pi} \left(1 - \frac{|\beta|}{2\lambda}\right) e^{i\eta\beta}, \quad (\eta \text{ constant}).$$

Note that for this particular function we have

$$(1.8) \quad \begin{aligned} \int_{-2\lambda}^{2\lambda} H(\beta) e^{-i\beta x} d\beta &= \frac{1}{\pi} \int_{-2\lambda}^{2\lambda} \left(1 - \frac{|\beta|}{2\lambda}\right) e^{i\beta(\eta-x)} d\beta = \\ &= \frac{2}{\pi} \int_0^{2\lambda} \left(1 - \frac{\beta}{2\lambda}\right) \cos \beta(\eta-x) d\beta = \frac{2}{\pi} \frac{\sin^2 \lambda(\eta-x)}{\lambda(\eta-x)^2}. \end{aligned}$$

Defining

$$(1.9) \quad K_{\lambda}(\zeta) := \frac{\sin^2 \lambda \zeta}{\pi \lambda \zeta^2}$$

we thus have

$$(1.10) \quad \begin{aligned} \int_{-2\lambda}^{2\lambda} H(\beta) \Delta(1+\alpha+\beta i) d\beta &= \\ &= 2 \int_0^{\infty} e^{-\alpha x} K_{\lambda}(\eta-x) L(x) dx - 2A \int_0^{\infty} e^{-\alpha x} K_{\lambda}(\eta-x) dx. \end{aligned}$$

It is clear that, if $\alpha \downarrow 0$, then the left hand side of (1.10) tends to

$$(1.11) \quad \int_{-2\lambda}^{2\lambda} H(\beta) r(\beta) d\beta.$$

Since

$$(1.12) \quad \int_0^{\infty} K_{\lambda}(\eta-x) dx = \int_0^{\infty} \frac{\sin^2 \lambda(\eta-x)}{\pi \lambda(\eta-x)^2} dx = \frac{1}{\pi} \int_{-\infty}^{\lambda\eta} \frac{\sin^2 u}{u^2} du$$

we have by a simple (Abelian) continuity theorem for Laplace transforms (see DOETSCH [2; p. 156])

$$(1.13) \quad \lim_{\alpha \downarrow 0} \int_0^{\infty} e^{-\alpha x} K_{\lambda}(\eta-x) dx = \int_0^{\infty} K_{\lambda}(\eta-x) dx$$

and hence

$$\begin{aligned}
 (1.14) \quad & \lim_{\alpha \downarrow 0} \int_0^{\infty} e^{-\alpha x} K_{\lambda}(\eta-x)L(x) dx = \\
 & \dots \text{ (use the fact: } K_{\lambda}(\eta-x) \geq 0 \text{) } \dots \\
 & = \int_0^{\infty} K_{\lambda}(\eta-x)L(x) dx = \\
 & = \frac{1}{2} \int_{-2\lambda}^{2\lambda} H(\beta)r(\beta) d\beta + \frac{A}{\pi} \int_{-\infty}^{\lambda\eta} \frac{\sin^2 u}{u^2} du.
 \end{aligned}$$

Now we let $\eta \rightarrow \infty$. Then (by RLL)

$$(1.15) \quad \lim_{\eta \rightarrow \infty} \int_{-2\lambda}^{2\lambda} H(\beta)r(\beta) d\beta = 0$$

and

$$(1.16) \quad \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} du = \pi$$

so that

$$(1.17) \quad \lim_{\eta \rightarrow \infty} \int_0^{\infty} K_{\lambda}(\eta-x)L(x) dx = A$$

which may also be written as

$$(1.18) \quad \lim_{\eta \rightarrow \infty} \int_{-\infty}^{\lambda\eta} L(\eta - \frac{u}{\lambda}) \frac{\sin^2 u}{u^2} du = \pi A.$$

1 Choose $\eta \geq \frac{1}{\sqrt{\lambda}}$ so that $\lambda\eta \geq \sqrt{\lambda}$ and hence

$$(1.19) \quad \int_{-\infty}^{\lambda\eta} L(\eta - \frac{u}{\lambda}) \frac{\sin^2 u}{u^2} du \geq \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} L(\eta - \frac{u}{\lambda}) \frac{\sin^2 u}{u^2} du \geq$$

... (if $t_2 \geq t_1$ then $F(t_2) \geq F(t_1)$ so that

$e^{t_2} L(t_2) \geq e^{t_1} L(t_1)$ and hence $L(t_2) \geq L(t_1) e^{t_1 - t_2}$) ...

$$\geq \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} L(\eta - \frac{1}{\sqrt{\lambda}}) \exp(-\frac{1}{\sqrt{\lambda}} + \frac{u}{\lambda}) \frac{\sin^2 u}{u^2} du \geq$$

$$\geq L(\eta - \frac{1}{\sqrt{\lambda}}) \exp(-\frac{2}{\sqrt{\lambda}}) \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2 u}{u^2} du.$$

From this it follows that

$$(1.20) \quad L(\eta - \frac{1}{\sqrt{\lambda}}) \leq \left(\int_{-\infty}^{\lambda\eta} L(\eta - \frac{u}{\lambda}) \frac{\sin^2 u}{u^2} du \right) e^{\frac{2}{\sqrt{\lambda}}} \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2 u}{u^2} du \right\}^{-1}.$$

Combining this with (1.18) we thus obtain

$$(1.21) \quad \limsup_{\eta \rightarrow \infty} L(\eta) \leq \pi A \exp\left(\frac{2}{\sqrt{\lambda}}\right) \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2 u}{u^2} du \right\}^{-1}.$$

By letting $\lambda \rightarrow \infty$ we obtain from this

$$(1.22) \quad \limsup_{\eta \rightarrow \infty} L(\eta) \leq A.$$

2 We have just seen that $L(\eta)$ is bounded at $+\infty$, i.e.

$$(1.23) \quad (0 \leq) L(\eta) \leq G, \quad (\eta \geq \eta_0).$$

For $\eta > \frac{1}{\sqrt{\lambda}}$ we thus have

$$(1.24) \quad \begin{aligned} & \int_{-\infty}^{\lambda\eta} L\left(\eta - \frac{u}{\lambda}\right) \frac{\sin^2 u}{u^2} du = \\ & = \left\{ \int_{-\infty}^{-\sqrt{\lambda}} + \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} + \int_{\sqrt{\lambda}}^{\lambda\eta} \right\} L\left(\eta - \frac{u}{\lambda}\right) \frac{\sin^2 u}{u^2} du \leq \\ & \leq \int_{-\infty}^{-\sqrt{\lambda}} \frac{G}{u^2} du + \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} L\left(\eta - \frac{u}{\lambda}\right) \frac{\sin^2 u}{u^2} du + \int_{\sqrt{\lambda}}^{\infty} \frac{G}{\sqrt{\lambda} u^2} du \leq \\ & \dots \text{ (see the note in part } \underline{1} \text{) } \dots \\ & \leq \frac{2G}{\sqrt{\lambda}} + \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} L\left(\eta + \frac{1}{\sqrt{\lambda}}\right) \exp\left(-\frac{1}{\sqrt{\lambda}} + \frac{u}{\lambda}\right) \frac{\sin^2 u}{u^2} du \leq \\ & \leq \frac{2G}{\sqrt{\lambda}} + L\left(\eta + \frac{1}{\sqrt{\lambda}}\right) \exp\left(\frac{2}{\sqrt{\lambda}}\right) \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2 u}{u^2} du. \end{aligned}$$

Combining this with (1.18) we easily obtain that

$$(1.25) \quad \liminf_{\eta \rightarrow \infty} L(\eta) \geq A$$

and the desired result

$$(1.26) \quad \lim_{\eta \rightarrow \infty} L(\eta) = A \quad \text{or} \quad \lim_{x \rightarrow \infty} \frac{F(x)}{e^x} = A$$

follows from 1 and 2. \square

REMARK. The original proof of Ikehara's theorem was based on Wiener's general Tauberian theorem. The proof above is Doetsch's version of Bochner's simplification of Ikehara's proof.

2. BOCHNER'S THEOREM

Without proof we state Bochner's

THEOREM 2.1. (1933, BOCHNER [1]) Let $\alpha : [0, \infty) \rightarrow \mathbf{R}$ be non-decreasing and let

$$(2.1) \quad f(s) = \int_0^{\infty} e^{-sx} d\alpha(x)$$

be convergent for $\operatorname{Re} s > 1$.

If

$$(2.2) \quad f(s) - \frac{A}{s-1} \rightarrow g(t), \quad \sigma \downarrow 1$$

uniformly in every finite interval $-\lambda \leq t \leq \lambda$, then

$$(2.3) \quad \lim_{x \rightarrow \infty} \alpha(x) e^{-x} = A.$$

As an immediate consequence of this theorem we have the following useful

THEOREM 2.2. Let $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots \rightarrow \infty$ and $a_n \geq 0, \forall n \in \mathbf{N}$ and suppose that the series

$$(2.4) \quad f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

is convergent for $\operatorname{Re} s > 1$.

If

$$(2.5) \quad f(s) - \frac{A}{s-1}$$

has a continuous extension up to $\sigma \geq 1$, then

$$(2.6) \quad \lim_{x \rightarrow \infty} e^{-x} \sum_{\lambda_n \leq x} a_n = A.$$

PROOF. See BOCHNER [1]. \square

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CHAPTER 14

THE PRIME NUMBER THEOREM

1. NUMBER-THEORETICAL PRELIMINARIES

DEFINITION. By $\pi(x)$ we denote the number of prime numbers not exceeding $x \in \mathbb{R}$. Furthermore we set

$$(1.1) \quad \theta(x) := \sum_{p \leq x} \log p, \quad (x > 0)$$

and

$$(1.2) \quad \psi(x) := \sum_{m=1}^{\infty} \theta(x^{\frac{1}{m}}), \quad (x > 0).$$

LEMMA 1. If $x > 0$, then

$$(1.3) \quad \pi(x) < x$$

and

$$(1.4) \quad \theta(x) \leq \pi(x) \log x.$$

PROOF. If $0 < x < 2$ then $\pi(x) = 0$ and if $x \geq 2$ then $\pi(x) \leq x-1 < x$. From (1.1) it is clear that

$$(1.5) \quad \theta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x. \quad \square$$

LEMMA 2.

$$(1.6) \quad \lim_{x \rightarrow \infty} \frac{\psi(x) - \theta(x)}{x} = 0.$$

PROOF. If $x \geq 4$ and $N := \lceil \frac{\log x}{\log 2} \rceil$ then

$$(1.7) \quad 0 \leq \psi(x) - \theta(x) = \sum_{m=2}^N \theta(x^{\frac{1}{m}}) \leq (N-1)\theta(x^{\frac{1}{2}}) < \\ < N \cdot \theta(x^{\frac{1}{2}}) \leq \frac{\log x}{\log 2} \pi(x^{\frac{1}{2}}) \log x^{\frac{1}{2}} < \frac{\log^2 x}{2 \log 2} x^{\frac{1}{2}}$$

and the lemma follows easily. \square

LEMMA 3. Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers. Define $P(x)$ as the number of elements of this sequence not exceeding x and let

$$(1.8) \quad L(x) := \sum_{a_k \leq x} \log a_k.$$

If $\frac{L(x)}{x}$ is bounded then

$$(1.9) \quad \lim_{x \rightarrow \infty} \left\{ \frac{P(x) \log x}{x} - \frac{L(x)}{x} \right\} = 0.$$

PROOF. For $1 < y < x$ we have

$$(1.10) \quad L(x) - L(y) = \sum_{y < a_k \leq x} \log a_k \geq \{P(x) - P(y)\} \log y$$

or

$$(1.11) \quad P(x) \leq P(y) + \frac{L(x) - L(y)}{\log y}.$$

Clearly $0 \leq L(x) \leq P(x) \log x$ and $P(x) \leq x$ so that

$$(1.12) \quad \begin{aligned} \frac{L(x)}{x} &\leq \frac{P(x) \log x}{x} \leq \left\{ P(y) + \frac{L(x) - L(y)}{\log y} \right\} \frac{\log x}{x} \leq \\ &\leq \left(y + \frac{L(x)}{\log y} \right) \frac{\log x}{x} = \frac{y}{x} \log x + \frac{L(x)}{x} \frac{\log x}{\log y}. \end{aligned}$$

Now choose $y = x^{1-\rho(x)}$ with

$$(1.13) \quad \rho(x) = \frac{(\log \log x)^2}{\log x}, \quad (x > e).$$

Then $0 < \rho(x) < 1$ so that $1 < y < x$. Hence

$$(1.14) \quad \frac{L(x)}{x} \leq \frac{P(x) \log x}{x} \leq x^{-\rho} \log x + \frac{L(x)}{x} \frac{1}{1-\rho}$$

or

$$(1.15) \quad 0 \leq \frac{P(x) \log x}{x} - \frac{L(x)}{x} \leq x^{-\rho} \log x + \frac{L(x)}{x} \left\{ \frac{1}{1-\rho} - 1 \right\}.$$

Since

$$(1.16) \quad x^{-\rho} \log x = \exp\{-(\log \log x)^2 + \log \log x\},$$

$$(1.17) \quad \frac{L(x)}{x} \leq G \quad \text{for some constant } G,$$

and

$$(1.18) \quad \lim_{x \rightarrow \infty} \rho(x) = 0$$

the lemma follows. \square

LEMMA 4.

$$(1.19) \quad (0 \leq) \frac{\theta(x)}{x} < 4 \log 2, \quad (x > 0).$$

PROOF. Since $2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k}$, ($n \in \mathbb{N}$), we have

$$(1.20) \quad \binom{2n}{n} < 2^{2n}, \quad (n \in \mathbb{N}).$$

Note that the binomial coefficient $\binom{2n}{n}$ is a positive integer which is divisible by all prime numbers p satisfying $n < p \leq 2n$. Hence

$$(1.21) \quad \prod_{n < p \leq 2n} p \leq \binom{2n}{n} < 2^{2n}$$

or

$$(1.22) \quad \theta(2n) - \theta(n) < 2n \log 2.$$

For $n = 2^{m-1}$ we get

$$(1.23) \quad \theta(2^m) - \theta(2^{m-1}) < 2^m \log 2$$

and it follows that

$$(1.24) \quad \begin{aligned} \theta(2^m) &= \{\theta(2^m) - \theta(2^{m-1})\} + \{\theta(2^{m-1}) - \theta(2^{m-2})\} + \dots + \{\theta(2^1) - \theta(2^0)\} < \\ &< 2^m \log 2 + 2^{m-1} \log 2 + \dots + 2^1 \log 2 = (2^{m+1} - 1) \log 2 < 2^{m+1} \log 2. \end{aligned}$$

Now observe that for every $x \geq 1$ there exists a positive integer m such that $2^{m-1} \leq x < 2^m$, so that

$$(1.25) \quad \theta(x) \leq \theta(2^m) < 2^{m+1} \log 2 = (4 \log 2) 2^{m-1} \leq (4 \log 2)x$$

and the lemma follows. \square

In Lemma 3 let $\{a_k\}_{k=1}^{\infty}$ be the sequence of prime numbers: $2, 3, 5, 7, \dots$. Then $P(x) = \pi(x)$ and $L(x) = \theta(x)$. We have just shown that $\theta(x)/x$ is bounded, so that by Lemma 3

$$(1.26) \quad \lim_{x \rightarrow \infty} \left\{ \frac{\pi(x) \log x}{x} - \frac{\theta(x)}{x} \right\} = 0$$

and combining this with (1.6) we obtain

LEMMA 5.

$$(1.27) \quad \lim_{x \rightarrow \infty} \left\{ \frac{\pi(x) \log x}{x} - \frac{\psi(x)}{x} \right\} = 0.$$

Hence, in order to prove the prime number theorem, i.e.

$$(1.28) \quad \lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1$$

we may just as well prove that

$$(1.29) \quad \lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1.$$

LEMMA 6.

$$(1.30) \quad \psi(x) = \sum_{p^m \leq x} \log p. \quad (m \in \mathbf{N}; p \text{ prime})$$

PROOF.

$$(1.31) \quad \psi(x) = \sum_{m=1}^{\infty} \theta\left(\frac{x}{m}\right) = \sum_{m=1}^{\infty} \sum_{p \leq x^{1/m}} \log p = \sum_{p^m \leq x} \log p.$$

REMARK. From this lemma it is clear that ψ is a step function having its jumps at the points $x = p^m$, (p prime; $m \in \mathbf{N}$).

DEFINITION.

$$(1.32) \quad \Lambda(n) := \psi(n) - \psi(n-1), \quad (n \in \mathbf{N}).$$

LEMMA 7.

$$(1.33) \quad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \\ 0 & \text{if } n \neq p^m \end{cases} \quad (m \geq 1).$$

PROOF. This is clear from Lemma 5. \square

LEMMA 8.

$$(1.34) \quad \sum_{m|k} \Lambda(m) = \log k, \quad (k \in \mathbf{N}).$$

PROOF. In the sum in the left hand side we only need to consider those divisors of k which have the form p^m , $m \geq 1$.

The lemma is clearly true for $k = 1$.

Therefore suppose that $k \geq 2$ and that $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ is the canonical prime decomposition of k . Then

$$(1.35) \quad \sum_{m|k} \Lambda(m) = \sum_{i=1}^r \sum_{\delta=1}^{\alpha_i} \Lambda(p_i^\delta) = \sum_{i=1}^r \alpha_i \log p_i = \log k.$$

2. SOME FUNDAMENTAL FACTS ABOUT $\zeta(s)$

The series $\sum_{n=1}^{\infty} n^{-s}$ is absolutely convergent for $\sigma > 1$ and defines a regular function (denoted by $\zeta(s)$) on this half plane.

As usual n^s , $s \in \mathbb{C}$, is defined as $\exp(s \log n)$, where $\log n = \int_1^n \frac{dx}{x}$.

If $\sigma > 1$ we may write

$$\begin{aligned}
 (2.1) \quad \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} = \int_{1-0}^{\infty} x^{-s} d[x] = \\
 &= [x]x^{-s} \Big|_{1-0}^{\infty} - \int_1^{\infty} [x] dx^{-s} = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx = \\
 &= s \int_1^{\infty} \frac{dx}{x^s} - s \int_1^{\infty} \frac{x-[x]}{x^{s+1}} dx = \frac{s}{s-1} - s \int_1^{\infty} \frac{x-[x]}{x^{s+1}} dx.
 \end{aligned}$$

Since $x-[x]$ is bounded, the last integral represents a regular function on $\sigma > 0$, so that $\zeta(s)$ has an analytic continuation up to $\sigma > 0$, having a simple pole at $s = 1$ with residue 1.

LEMMA 2.1. The series $\sum_{n=1}^{\infty} \Lambda(n)n^{-s}$ is absolutely convergent for $\sigma > 1$ and satisfies

$$(2.2) \quad \zeta(s) \sum_{n=1}^{\infty} \Lambda(n)n^{-s} = -\zeta'(s), \quad (\sigma > 1).$$

PROOF. The absolute convergence of the series for $\sigma > 1$ follows from the estimate $0 \leq \Lambda(n) \leq \log n$.

Now observe that (by absolute convergence) for $\sigma > 1$

$$\begin{aligned}
 (2.3) \quad \zeta(s) \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s} &= \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right) \left(\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s} \right) = \\
 &= \sum_{m,n=1}^{\infty} \frac{\Lambda(m)}{(mn)^s} = \sum_{k=1}^{\infty} \frac{1}{k^s} \left(\sum_{m|k} \Lambda(m) \right) = \sum_{k=1}^{\infty} \frac{\log k}{k^s} = -\zeta'(s). \square
 \end{aligned}$$

LEMMA 2.2. On $\sigma > 1$ we have $\zeta(s) \neq 0$ so that

$$(2.4) \quad \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}, \quad (\sigma > 1).$$

PROOF. Suppose that $\zeta(s)$ has a zero at $s_0 = \sigma_0 + it_0$ (where $\sigma_0 > 1$) of order $\alpha (\geq 1)$. Then the product

$$(2.5) \quad \zeta(s) \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

has a zero of order $\beta \geq \alpha$ at $s = s_0$.

Since $\zeta(s)$ has a zero of order α at $s = s_0$, $\zeta'(s)$ has a zero of

order $\alpha-1$ at $s = s_0$. This leads to the palpable contradiction $\alpha \leq \beta = \alpha-1$. \square

REMARK. The fact that $\zeta(s) \neq 0$ also follows from Euler's identity

$$(2.6) \quad \zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}, \quad (\sigma > 1).$$

LEMMA 2.3. For the analytic continuation of $\zeta(s)$ we have

$$(2.7) \quad \zeta(s) \neq 0 \quad \text{on} \quad \sigma = 1.$$

PROOF. If $\sigma > 1$ then we may write (see (2.6))

$$(2.8) \quad \begin{aligned} \zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \exp\left(\sum_{m=1}^{\infty} \frac{1}{mp^{ms}}\right) = \\ &= \exp\left(\sum_{p,m} \frac{1}{m} \exp(-ms \log p)\right). \end{aligned}$$

Consequently

$$(2.9) \quad |\zeta(\sigma+it)| = \exp\left(\sum_{p,m} \frac{\cos(mt \log p)}{mp^{m\sigma}}\right)$$

so that

$$(2.10) \quad \begin{aligned} \zeta^3(1+\epsilon) |\zeta(1+\epsilon+it)|^4 |\zeta(1+\epsilon+2it)| &= \\ &= \exp\left(\sum_{p,m} \frac{3+4\cos(mt \log p) + \cos(2mt \log p)}{mp^{m(1+\epsilon)}}\right). \end{aligned}$$

Now observe that for every $\phi \in \mathbb{R}$ one has

$$(2.11) \quad 3+4\cos \phi + \cos 2\phi = 3+4\cos \phi + 2\cos^2 \phi - 1 = 2(1+\cos \phi)^2 \geq 0$$

so that

$$(2.12) \quad \zeta^3(1+\epsilon) |\zeta(1+\epsilon+it)|^4 |\zeta(1+\epsilon+2it)| \geq 1$$

which may also be written as

$$(2.13) \quad |\zeta(1+\epsilon+it)|^4 \geq \frac{1}{\zeta^3(1+\epsilon) |\zeta(1+\epsilon+2it)|}.$$

Since, for $0 < \epsilon < 1$,

$$(2.14) \quad (0 <) \zeta(1+\epsilon) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{1+\epsilon}} < 1 + \int_1^{\infty} \frac{dx}{x^{1+\epsilon}} = 1 + \frac{1}{\epsilon} < \frac{2}{\epsilon}$$

the right hand side of (2.13) is larger than $\epsilon^3/8|\zeta(1+\epsilon+2it)|$, and it follows that

$$(2.15) \quad \left| \frac{\zeta(1+\epsilon+it)}{\epsilon} \right|^4 \geq \frac{1}{8\epsilon|\zeta(1+\epsilon+2it)|}.$$

Since $\zeta(s)$ is analytic at $s = 1+2it$, $t \neq 0$, we clearly have

$$(2.16) \quad \lim_{\epsilon \downarrow 0} \epsilon|\zeta(1+\epsilon+2it)| = 0.$$

Combining this with (2.15) we obtain

$$(2.17) \quad \lim_{\epsilon \downarrow 0} \left| \frac{\zeta(1+\epsilon+it)}{\epsilon} \right| = \infty, \quad (t \neq 0).$$

If $\zeta(1+it) = 0$ for some $t \neq 0$, then

$$(2.18) \quad \left| \frac{\zeta(1+\epsilon+it)}{\epsilon} \right| = \left| \frac{\zeta(1+\epsilon+it) - \zeta(1+it)}{\epsilon} \right|$$

which (because of the regularity of $\zeta(s)$ on the line $\sigma = 1$, $s \neq 1$) tends to $|\zeta'(1+it)|$ as $\epsilon \downarrow 0$. However, this contradicts (2.17) so that we have to conclude that $\zeta(1+it) \neq 0$ for all $t \neq 0$. \square

LEMMA 2.4. The function $-\frac{\zeta'(s)}{s\zeta(s)}$ is regular on $\sigma \geq 1$, except at $s = 1$ where it has a simple pole with residue 1.

PROOF. From the previous analysis it is clear that the function under consideration is regular on $\sigma \geq 1$, $s \neq 1$.

The Laurent expansion of $\zeta(s)$ around $s = 1$ reads as follows

$$(2.19) \quad \zeta(s) = \frac{1}{s-1} + c_0 + c_1(s-1) + \dots, \quad (0 < |s-1| < 1)$$

from which it follows by straightforward computation that

$$(2.20) \quad -\frac{\zeta'(s)}{s\zeta(s)} = \frac{1}{s-1} - (c_0+1) + \dots$$

for all s in some neighborhood of 1 ($s \neq 1$, of course). \square

LEMMA 2.5. The function

$$(2.21) \quad -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1}$$

is regular on $\sigma \geq 1$.

PROOF. This is a restatement of Lemma 2.4. \square

LEMMA 2.6.

$$(2.22) \quad -\frac{\zeta'(s)}{s\zeta(s)} = \int_0^\infty e^{-st} \psi(e^t) dt, \quad (\sigma > 1).$$

PROOF. For $\sigma > 1$ we have by Lemma 2.2

$$(2.23) \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

whereas the series may be written as

$$(2.24) \quad \begin{aligned} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{\psi(n) - \psi(n-1)}{n^s} = \int_{+0}^{\infty} \frac{1}{x^s} d\psi(x) = \\ &= \frac{\psi(x)}{x^s} \Big|_{+0}^{\infty} - \int_{+0}^{\infty} \psi(x) dx^{-s} = s \int_1^{\infty} \frac{\psi(x)}{x^{s+1}} dx = \\ &= s \int_0^{\infty} e^{-st} \psi(e^t) dt \end{aligned}$$

proving the lemma. \square

3. THE PRIME NUMBER THEOREM

THEOREM 3.1.

$$(3.1) \quad \lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1.$$

PROOF. In Ikehara's theorem take

$$(3.2) \quad F(t) := \psi(e^t), \quad (t \geq 0).$$

Clearly $F(t)$ is non-negative and non-decreasing and the Laplace transform of $F(t)$ is $-\zeta'(s)/s\zeta(s)$.

Since $-\zeta'(s)/s\zeta(s) - 1/(s-1)$ is regular on $\sigma \geq 1$, all conditions in Ikehara's theorem are satisfied so that

$$(3.3) \quad 1 = \lim_{t \rightarrow \infty} \frac{\psi(e^t)}{e^t} = \lim_{x \rightarrow \infty} \frac{\psi(x)}{x}. \quad \square$$

Combining (3.3) with (1.27) we obtain

(The Prime Number) THEOREM 3.2.

$$(3.4) \quad \lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1.$$

In the above derivation of the Prime Number Theorem the analytic behavior of $-\zeta'(s)/s\zeta(s) - 1/(s-1)$ turned out to be of crucial importance. The analyticity of this function on $\sigma \geq 1$ was obtained from the fact that $\zeta(s) \neq 0$ on $\sigma \geq 1$, $s \neq 1$. It was easy to show that $\zeta(s) \neq 0$ on $\sigma > 1$, so that the PNT is mainly a consequence of the fact that $\zeta(s) \neq 0$ on $\sigma = 1$.

We conclude by showing that this property of $\zeta(s)$ is also necessary for the PNT.

Indeed, if (3.4) holds, then (by Lemma 5) $\lim_{x \rightarrow \infty} \psi(x)/x = 1$ or $\lim_{t \rightarrow \infty} \psi(e^t)/e^t = 1$. Setting $F(t) := \psi(e^t)$, we obtain from Theorem 0.1 that the Laplace transform of $F(t)$ has no poles on $\sigma = 1$, $s \neq 1$. Hence, $-\zeta'(s)/s\zeta(s)$ has no poles on $\sigma = 1$, $s \neq 1$, so that (by a simple function theoretic argument) $\zeta(s)$ cannot have any zeros on $\sigma = 1$.

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